Answers exam Complex Functions 2010.

1. Write $f = u+iv$. It is given that $u(x,y) = \frac{1}{2}\log(x^2+y^2)$. Using the Cauchy Riemann equations we notice

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{y}{x^2+y^2}$$

The first equation gives us $v(x,y) = v(x,0) + \arctan(y/x)$. Plugging this in the second equation gives

$$\frac{\partial}{\partial x}\{v(x,0)\} - \frac{y}{x^2+y^2} = -\frac{y}{x^2+y^2}$$

Hence $v(x,0)$ is a constant function. Since $f(1) = 0$ we conclude that $v(x,0)$ is identically zero. Thus $v(x,y) = \arctan(y/x)$.

2. The fastest way to calculate the convergence radius is by using the ratiotest

$$\lim_{n\to\infty} \frac{(n+1)^{d-1}}{n^{d-1}} = \left(\lim_{n\to\infty} 1 + \frac{1}{n}\right)^{d-1} = 1.$$

We have $f_1(z) = (1-z)^{-1}$ so the statement is trivial for this case. Assume the statement is true for d. We notice (tacitly using Thm 5.1, page 72) that

$$\begin{aligned} f_{d+1}(z) = zf_d'(z) &= z\frac{p_d'(z)(1-z)^d + p_d(z)(d+1)(1-z)^{d-1}}{(1-z)^{2d}} \\ &= \frac{z(1-z)p_d'(z) + (d+1)zp_d(z)}{(1-z)^{d+1}} \end{aligned}$$

and the polynomial in the numerator indeed has degree at most d. We know that $(1-z)^d f_d(z)$ is a polynomial of degree at most $d-1$. Hence its d^{th} coefficient is zero. We can find the coefficients of $(1-z)^d$ by Newtons binomial theorem:

$$(1-z)^d = \sum_{n=0}^{d} \binom{d}{n} (-1)^n z^n.$$

Now using the expression for the product of two series we arrive at

$$\sum_{n=0}^{d} \binom{d}{n} (-1)^n n^{d-1} = (-1)^d \sum_{n=0}^{d} \binom{d}{d-n} (-1)^{d-n} n^{d-1} = 0$$

3. We know that $1/(z^n+1)$ has simple poles in the points $e^{\frac{\pi i}{n}} e^{\frac{2\pi i k}{n}}$, $k = 0, 1, \ldots, n-1$. We parameterize our chain γ by the following curves:

$$\begin{cases} \gamma_1(t) = t & 0 \le t \le R \\ \gamma_2(t) = Re^{it} & 0 \le t \le \frac{2\pi}{n} \\ \gamma_3(t) = te^{\frac{2\pi i}{n}} & 0 \le t \le R \text{(reverse direction)} \end{cases}$$

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First we estimate the integral over γ_2:

$$\lim_{R\to\infty}\left|\int_{\gamma_2}\frac{1}{z^n+1}dz\right| \leq \lim_{R\to\infty}\int_0^{2\pi/n}\frac{Rd\phi}{|R^n e^{int}+1|} \leq \lim_{R\to\infty}\frac{2\pi R}{n(R^n-1)} = 0$$

Hence the residue theorem gives us

$$\int_0^\infty \frac{dx}{x^n+1} + 0 - \int_0^\infty \frac{e^{\frac{2\pi i}{n}}dx}{e^{2\pi i}x^n+1} = \int_\gamma \frac{dz}{z^n+1} = 2\pi i\text{Res}(\frac{1}{1+z^n}, e^{\frac{\pi i}{n}})$$
$$= \frac{2\pi i}{n(e^{\frac{\pi i}{n}})^{n-1}} = \frac{2\pi i}{-ne^{\frac{-\pi i}{n}}}$$

We conclude that

$$\int_0^\infty \frac{dx}{x^n+1} = \frac{2\pi i}{-ne^{\frac{-\pi i}{n}}(1-e^{\frac{2\pi i}{n}})} = \frac{\pi/n}{\sin \pi/n}$$

4. Let U be a connected open set and let f be a function which is analytic on U. Let us assume that there exists a point $z_0 \in U$ with $|f(z_0)| \geq |f(z)|$ for all $z \in U$. Now assume f is not locally constant at z_0. Then f is an open mapping in a neighborhood of z_0. Thus there exists an open disk centered at $f(z_0)$ which is a subset of $f(U)$. But then $f(U)$ contains points that have a larger distance to the origin than $f(z_0)$. We conclude that f must be locally constant at z_0. By analytic continuation f must be constant on U.

5. Define the following function $F : \mathbb{C} \to \mathbb{C}$:

$$F(z) = \begin{cases} f(z) & \text{for } \text{Re}(z) > 0 \\ f(z+n)\prod_{k=0}^{n-1} g(z+k) & \text{for } -n < \text{Re}(z) \leq -n+1 \end{cases}$$

Clearly f is analytic on $\{z \in \mathbb{C} | \text{Re}(z) \notin \mathbb{Z}_{\leq 0}\}$. Suppose $\text{Re}(z_0) = -n \in \mathbb{Z}_{\leq 0}$. Denote by D a disk with radius < 1 centered at z_0. We notice that for $z \in D$ with $\text{Re}(z) > -n$ we have

$$F(z) = f(z+n)\prod_{k=0}^{n-1} g(z+k) = f(z+n+1)\prod_{k=0}^{n} g(z+k)$$

We conclude that F is equal to the analytic function $f(z+n+1)\prod_{k=0}^{n} g(z+k)$ on D. Thus F is analytic in z_0. We conclude that F is analytic on $\mathbb{C}$.

6. Let $f \neq 0$ be such a function. We can write $f(z) = a_n z^n g(z)$ for some $n \geq 0$ and an analytic function $g : \mathbb{C} \to \mathbb{C}$ satisfying $g(0) = 1$ and $a_n \neq 0$. We notice using the residue theorem that

$$|a_n| = \left|\frac{1}{2\pi i}\int_{C_{|z|}} \frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(|z|e^{i\phi})|}{|z|^n}d\phi = \frac{|f(|z|)|}{|z|^n}$$

Thus $|f(|z|)| \geq |a_n||z|^n$. This implies $|g(z)| \geq 1$. Hence $1/g(0)$ is a maximum for the analytic function $1/g$. The maximum principle (or Liouville's Theorem) implies that g is constant. We conclude that functions of the form $f : \mathbb{C} \to \mathbb{C}$, $f(z) = a_n z^n$, are the only functions that satisfy the required properties.

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