ENDTERM COMPLEX FUNCTIONS

JUNE 29 2011, 9:00-12:00

- Put your name and studentnummer on every sheet you hand in.
- When you use a theorem, show that the conditions are met.

Exercise 1. ***(5 pt)*** Let $U = \{z \in \mathbb{C} : z \neq x,\ x \leq 0\}$. Prove that any continuous closed path γ in U is homologous to 0 in U.

Exercise 2. ***(10 pt)*** Show that equation $z^6 + 4z^2 = 1$ has exactly two simple roots with $|z| < 1$.

Exercise 3. ***(10 pt)*** Let $a > b > 0$. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)}\, dx.$$

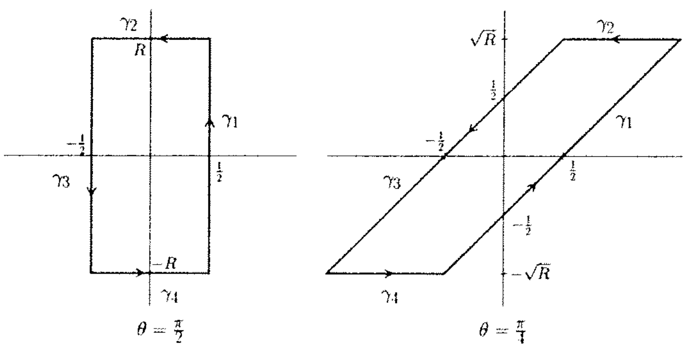
Exercise 4. ***(20 pt)*** Consider the function $f(z) = \dfrac{e^{i\pi z^2}}{\sin(\pi z)}$.

a. *(2 pt)* Find all singular points of f in $\mathbb{C}$ and determine their type. Compute $\mathrm{Res}_0 f$.

b. *(5 pt)* Show that $f(z) - f(z-1) = 2ie^{i\pi z(z-1)}$ for $z \neq 0, \pm 1, \pm 2, \ldots$.

c. *(3 pt)* Let γ be a closed path composed by four straight line segments $\gamma_k, k = 1,2,3,4$, connecting the points

$$\pm\frac{1}{2} \pm Re^{i\theta}, \quad 0 < \theta < \pi,$$



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and oriented counterclockwise. Let γ_1 and γ_3 be the right and left sides of γ, respectively. Prove that

$$\int_{\gamma_3} f(z)dz = -\int_{\gamma_1} f(z-1)dz.$$

d. (*5 pt*) Let γ_2 and γ_4 be the top and bottom sides of γ, respectively. Show for $\theta = \frac{\pi}{2}$ and $\frac{\pi}{4}$ that

$$\lim_{R\to\infty} \left| \int_{\gamma_2} f(z)dz + \int_{\gamma_4} f(z)dz \right| = 0.$$

e. (*5 pt*) Use the information obtained for $\theta = \frac{\pi}{2}$ and $\frac{\pi}{4}$ to evaluate the integrals

$$\int_{-\infty}^{\infty} \cos(\pi x^2)dx \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-\pi x^2} dx,$$

respectively.

Bonus Exercise. (*10 pt*) Let $f(z) = \dfrac{1}{z^2 \sin z}$.

a. (*5 pt*) Find poles of f and compute the residue of f at each of them.

b. (*5 pt*) Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Exercise 5. A shop has two identical machines, one of which is kept as a backup. A working machine fails after an exponential time of rate λ, in which case it is sent to the service department and replaced by the backup machine if the latter is working. The service department employs a technician that can repair only one machine at a time and takes an exponential time of rate μ to repair a machine. The repaired machine becomes the new backup.

(a) (8 pts.) Model this process as a birth-and-death process with state i= number of down machines. Determine the birth and death rates.

(b) (8 pts.) Determine the expected time until both machines are simultaneously in the service department.

(c) (8 pts.) Determine, in the long run, the proportion of time the shop has no machine in working order.

(d) (8 pts.) Write the 9 backward Kolmogorov equations, and observe that they form three sets of three coupled linear differential equations.

(e) (4 pts.) If $\lambda = \mu$, prove that $P_{00}(t) - P_{20}(t) = e^{-\lambda t}$.

Bonus problems

Only one of them may count for the grade

You can try both, but only the one with the highest grade will be considered

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Bonus 1. (6 pts.) *Alternative definition of Poisson processes:* Let $N(t)$ be a counting process of the form

$$N(t) = \max\{n : S_n \le t\}$$

where

$$S_n = \sum_{i=1}^{n} T_i$$

for iid $\mathrm{Exp}(\lambda)$ random variables T_i, $i \ge 1$. Prove that, indeed,

$$P\big(N(t) = n\big) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} .$$

Bonus 2. *Continuous-time Ehrenfest urn model:* N molecules are distributed between two urns and at successive $\mathrm{Exp}(\lambda)$ independent random times one molecule is chosen with uniform probability and displaced from its original urn to the other one. Let $X(t)$ = number of molecules in the first urn and denote $M(t) = E\big[X(t)\big]$.

(a) (3 pts.) Find the explicit expressions for $P(t,h)$ and $Q(t,h)$ in the following table:

$$X(t+h) = \begin{cases} X(t)+1 & \text{with probability } P(t,h) \\ X(t)-1 & \text{with probability } Q(t,h) \\ X(t) & \text{with probability } 1 - P(t,h) - Q(t,h) . \end{cases}$$

(b) (3 pts.) Prove that $M(t)$ satisfies the differential equation

$$\frac{dM(t)}{dt} = \lambda\Big[1 - \frac{2M(t)}{N}\Big] .$$