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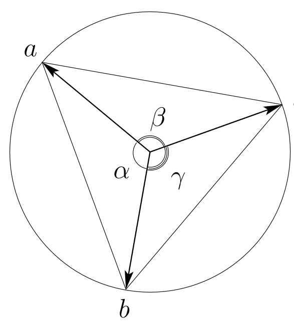
MIDTERM COMPLEX FUNCTIONS

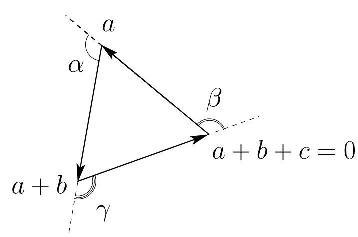
APRIL 18 2012, 9:00-12:00

Exercise 1 (*7 pt*) Let $a, b, c \in \mathbb{C}$ be located on the unit circle and let $a + b + c = 0$. Prove that the corresponding points are the vertices of an equilateral triangle.

Without loss of generality $a = 1$. Then $\mathrm{Im}(b) + \mathrm{Im}(c) = 0$. Since both b and c are on the circle this is only possible when $c = \overline{b}$ (the case $c = -b$ is ruled out). Thus $2\mathrm{Re}(b) = -1$, from which it follows that $\{b, c\} = \{e^{\pm 2\pi/3}\}$, and we conclude that abc forms an equilateral triangle.

Remark: One can also find a geometrical proof. Consider the three vectors from 0 to a, a to $a + b$ and from $a + b$ to $a + b + c$. Since $a + b + c = 0$ these vectors form a triangle, since they are unit vectors this triangle is equilateral. It follows from the picture below that $\alpha = \beta = \gamma = 2\pi/3$ and thus abc forms an equilateral triangle.





Exercise 2 (*10 pt*) Write the Cauchy-Riemann equations in polar coordinates (r, θ). Then show that the function $\log z = \log \rho + i\theta,\ z = re^{i\theta}$, is holomorphic in the region $r > 0,\ -\pi < \theta < \pi$.

We write $x = r\cos\phi$ and $y = r\sin\phi$ and $f = u + iv$ as usual. We notice using the chainrule and the Cauchy-Riemann equations that

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos(\phi) + \frac{\partial u}{\partial y}\sin(\phi) \\
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = -\frac{\partial u}{\partial y}\cos(\phi) + \frac{\partial u}{\partial x}\sin(\phi) \\
\frac{\partial u}{\partial \phi} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} = -\frac{\partial u}{\partial x}r\sin(\phi) + \frac{\partial u}{\partial y}r\cos(\phi) \\
\frac{\partial v}{\partial \phi} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \phi} = \frac{\partial u}{\partial y}r\sin(\phi) + \frac{\partial u}{\partial x}r\cos(\phi).
\end{aligned}$$

It follows that

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \phi} \text{ and } r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \phi}$$

The log as defined in the exercise obeys these equations (strictly speaking you should show that these are equivalent to the CR equations) and is thus holomorphic.

Exercise 3 (*10 pt*) Suppose $f : U \to \mathbb{C}$ is a non-constant holomorphic function on an open set $U \subset \mathbb{C}$ containing the closed unit disc $\overline{D(0,1)}$. Suppose that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$. Prove that the equation $f(z) = 0$ has a solution in the open unit disc $D(0,1)$.

By the maximum modulus principle $f(z)$ attains its (absolute) maximum (only) at the unit circle. However, if f does not have a root in the open unit disk then we can define $g : \overline{D(0;1)} \to \mathbb{C}$ by $g(z) = 1/f(z)$. Then by the maximum modulus principle g attains its maximum (only) at the unit circle. But then the minimum and maximum of f equal each other. Thus $|f| = 1$ and the maximum modulus principle implies that f is constant.

Exercise 4 (*8 pt*) Compute

$$\int_\gamma \frac{\sin z}{z^2}\,dz \quad \text{and} \quad \int_\gamma \frac{\cos z}{z^3}\,dz,$$

where γ is the unit circle $|z| = 1$ oriented counter-clockwise and traced once.

Using the extended Cauchy formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{|z-a|=r} \frac{f(z)}{(z-a)^{n+1}}dz$$

we get

$$
\begin{aligned}
\oint_\gamma \frac{\sin z}{z^2} dz &= 2\pi i \cos(0) = 2\pi i \\
\oint_\gamma \frac{\cos z}{z^3} dz &= 2\pi i(-\cos(0))/2! = -\pi i.
\end{aligned}
$$

Exercise 5 (*10 pt*) Suppose that a complex function f has a power series representation near the origin, i.e. there is a power series $\sum_{n=0}^{\infty} a_n z^n$ that converges absolutely to $f(z)$ in an open disc centered at $z = 0$.

(i) Assuming that $a_0 \neq 0$, prove that the function

$$g(z) = \frac{1}{f(z)}$$

also has has a power series representation near the origin.

Define $h(z) = 1 - f(z)/a_0$. Then $h(z)$ is analytic and this implies that

$$\sum_{n=1}^{\infty} |a_n||z|^n$$

coverges uniformly in some open neighborhood of $z = 0$. This means in particular that it is continous and thus the we can find an open neighborhood small enough such that it is smaller then 1. We may apply theorem 3.4 (p.66) to conclude that the composition of $1/(1-T)$ with $h(T)$ (notice this formal composition makes sence) converges uniformly to $1/(1-h(z)) = a_0/f(z)$ in this neighborhood. Thus g has a powerseries representation near the origin.

(ii) Derive explicit formulas for the coefficients b_0, b_1, b_2, and b_3 in the series $\sum_{n=0}^{\infty} b_n z^n$ representing the function g near the origin.

We have

$$
\begin{aligned}
a_0 g(z) &= 1 + h(z) + h(z)^2 + h(z)^3 + \dots \\
&= 1 + \left(-\frac{a_1}{a_0}\right) z + \left(-\frac{a_2}{a_0} + \frac{a_1^2}{a_0^2}\right) z^2 + \left(-\frac{a_3}{a_0} + 2\frac{a_1 a_2}{a_0^2} - \frac{a_1^3}{a_0^3}\right) z^3 + \dots
\end{aligned}
$$

Remark: alternatively one can use the correspondence holomorphic $\Leftrightarrow$ analytic and remark that g is a holomorphic function. For the second part one could find the coefficients by tayloring $1/f$.