

RETAKE COMPLEX FUNCTIONS

August 22 2012, 9:00-12:00

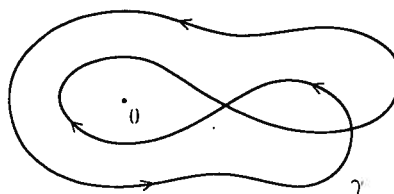
- Put your name and studentnummer on every sheet you hand in.
- When you use a theorem, show that the conditions are met.

Exercise 1. (20 pt) Let $k > 0$ with $k \neq 1$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$. Prove that the set

$$\{z \in \mathbb{C} : |z - \alpha| = k|z - \beta|\}$$

is a circle and find its center and radius.

Exercise 2. (10 pt) Compute $\int_{\gamma} \frac{e^z - e^{-z}}{z^4} dz$ along the path shown below.



Exercise 3. (20 pt) Let $w = f(z)$ be an analytic function in a neighbourhood of $z = 0$ with $f(0) = 0$ but $f'(0) \neq 0$.

a. (15 pt) Suppose that f has a series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

near $z = 0$. Derive explicit expressions for the coefficients b_n with $n \leq 5$ in the series representation of the inverse function $g = f^{-1}$ near $w = 0$:

$$g(w) = \sum_{n=0}^{\infty} b_n w^n.$$

b. (5 pt) Prove that

$$\arcsin w = w + \frac{1}{6}w^3 + \frac{3}{40}w^5 + O(|w|^7).$$

Turn the page!

Exercise 4. (20 pt) Let $f(z) = z^6 - 5z^4 + 10$.

a. (15 pt) Prove that f has

- (i) no zeroes with $|z| < 1$;
- (ii) 4 zeroes with $|z| < 2$;
- (iii) 6 zeroes with $|z| < 3$;

b. (5 pt) For cases (ii) and (iii), show that all zeroes are different.

Exercise 5. (30 pt) Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Consider the function

$$f(z) = \frac{\cos(\pi z)}{(z-a)^2 \sin(\pi z)}.$$

a. (10 pt) For integer $n \geq 0$, introduce the rectangular closed path γ_n passing through the points

$$\pm \left(n + \frac{1}{2} \right) \pm in$$

and oriented counter-clockwise. Prove by integral estimates that

$$\left| \int_{\gamma_n} f(z) dz \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

b. (10 pt) Find poles of f and compute the residue of f at each of them.

c. (5 pt) Show that

$$\int_{\gamma_n} f(z) dz = 2\pi i \left(-\frac{\pi}{\sin^2(\pi a)} + \sum_{k=-n}^n \frac{1}{\pi(k-a)^2} \right).$$

d. (5 pt) Compute $\sum_{n=-\infty}^{\infty} \frac{1}{(n-a)^2}$.

Bonus Exercise. (20 pt) Suppose that f is an entire function and for every point $a \in \mathbb{C}$ at least one of the coefficients c_n in the Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

vanishes. Prove that f is a polynomial.