MIDTERM COMPLEX FUNCTIONS SOLUTIONS

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Exercise 1.

a. It is well known that the geometric series converges for $|z| < 1$ and thus

$$\begin{aligned}\frac{1}{1-e^{i\phi}z} - \frac{1}{1-e^{-i\phi}z} &= \sum_{n=0}^{\infty} e^{in\phi}z^n - \sum_{n=0}^{\infty} e^{-in\phi}z^n \\ &= \sum_{n=0}^{\infty}(e^{in\phi} - e^{-in\phi})z^n = 2i\sum_{n=1}^{\infty}\sin(n\phi)z^n\end{aligned}$$

for $|z| < 1$. Thus $\rho \geq 1$. If $\rho > 1$ then the series should be analytic and hence continuous in $e^{i\phi}$, since this is not the case we must conclude that $\rho = 1$. Clearly our series equals a rational function on $|z| < 1$.

b. For $|z| < 1$ we have

$$\begin{aligned}-4\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sin(k\phi)\sin(n\phi - k\phi)\right)z^n &= \left(\frac{1}{1-e^{i\phi}z} - \frac{1}{1-e^{-i\phi}z}\right)^2 \\ &= \frac{1}{(1-e^{i\phi}z)^2} + \frac{1}{(1-e^{-i\phi}z)^2} - 2\frac{1}{(1-e^{i\phi}z)(1-e^{-i\phi}z)} \\ &= e^{-i\phi}\frac{d}{dz}\frac{1}{1-e^{i\phi}} + e^{i\phi}\frac{d}{dz}\frac{1}{1-e^{-i\phi}} - \frac{2}{e^{i\phi}-e^{-i\phi}}\left(\frac{e^{i\phi}}{1-e^{i\phi}z} - \frac{e^{-i\phi}}{1-e^{-i\phi}z}\right) \\ &= e^{-i\phi}\sum_{n=1}^{\infty}ne^{in\phi}z^{n-1} + e^{i\phi}\sum_{n=1}^{\infty}ne^{-in\phi}z^{n-1} \\ &\quad - \frac{2}{e^{i\phi}-e^{-i\phi}}\left(\sum_{n=0}^{\infty}e^{i(n+1)\phi}z^n - \sum_{n=0}^{\infty}e^{-i(n+1)\phi}z^n\right) \\ &= 2\sum_{n=0}^{\infty}\left((n+1)\cos(n\phi) - \frac{\sin(n\phi+\phi)}{\sin(\phi)}\right)z^n\end{aligned}$$

The fact that these two series coincide on an open set with accumulation point 0 implies that their coefficients are equal, and we are done.

c. We know that $\frac{2\pi}{n} \in (0, \pi)$ thus we may apply the formula from b.

$$\sum_{k=0}^{n} \sin^2\left(\frac{2\pi k}{n}\right) = \sum_{k=0}^{n} \sin\left(\frac{2\pi k}{n}\right) \sin\left(\frac{2\pi k}{n} - \frac{2\pi n}{n}\right) = \frac{1}{2}\left((n+1) - 1\right) = \frac{n}{2}.$$

Exercise 2. Define the polynomial $P : \mathbb{C} \to \mathbb{C}$

$$P(z) = \prod_{i=1}^{n} (z - z_i)$$

This is an analytic nonconstant function on $\mathbb{C}$. The Maximum Modulus Principle implies (see Corollary 1.4, p.92) that $|P(z)|$ attains its maximum over $\overline{D(0,1)}$ at a point on its boundary, i.e. on the unit circle.
Suppose that $|P(z)| \leq 1$ for all points z on the unit circle. Then we must have $|P(z)| < 1$ for all points $z \in D(0,1)$. But $|P(0)| = |z_1| \cdot |z_2| \ldots |z_n| = 1$, which is a contradiction, and we must conclude that there exists a point z on the unit circle such that $|z - z_1| \cdot |z - z_2| \cdots |z - z_n| = |P(z)| > 1$.

Exercise 3.
a. Let $z \in U$ and write $z = x + iy$ with x, y real. It follows that $(\mathrm{Re}f(z))^2 - (\mathrm{Im}f(z))^2 = x$ and $2(\mathrm{Re}f(z))(\mathrm{Im}f(z)) = y$. Since $y \neq 0$ we have $\mathrm{Re}f(z) \neq 0$ and we may write

$$(\mathrm{Re}f(z))^2 - \left(\frac{y}{2\mathrm{Re}f(z)}\right)^2 = x$$

We can write this as a quadratic equation:

$$((\mathrm{Re}f(z))^2)^2 - x(\mathrm{Re}f(z))^2 - \frac{y^2}{4}$$

It's solutions are

$$(\mathrm{Re}f(z))^2 = \frac{x \pm \sqrt{x^2 + y^2}}{2} = \pm\frac{|z| \pm \mathrm{Re}(z)}{2}.$$

Since $\mathrm{Re}f(z)$ is a real number we must take the plus sign. Also we find that

$$(\mathrm{Im}f(z))^2 = (\mathrm{Re}f(z))^2 - x = \frac{|z| + \mathrm{Re}(z)}{2} - \frac{2\mathrm{Re}(z)}{2} = \frac{|z| - \mathrm{Re}(z)}{2}.$$

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We conclude that there exist $\alpha, \beta : U \to \{-1, 1\}$ such that for all $z \in U \setminus \mathbb{R}$

$$\mathrm{Re} f(z) = \frac{\alpha(z)}{\sqrt{2}} \sqrt{|z| + \mathrm{Re}(z)} \text{ and } \mathrm{Im} f(z) = \frac{\beta(z)}{\sqrt{2}} \sqrt{|z| - \mathrm{Re}(z)}$$

b. Write $u(x, y) = \mathrm{Re} f(x + iy)$ and $v(x, y) = \mathrm{Im} f(x + iy)$. If the Cauchy Riemann equations are satisfied in some point then we may at least assume that α and β do not change sign in some open disc around that point. So

$$\frac{\partial u}{\partial x} = \frac{\alpha(x+iy)}{2\sqrt{2}} \left(\frac{x}{\sqrt{x^2+y^2}} + 1 \right) \frac{1}{\sqrt{\sqrt{x^2+y^2}+x}} = \frac{\alpha(x+iy)}{2\sqrt{2}} \frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}}$$

and

$$\frac{\partial v}{\partial y} = \frac{\beta(x+iy)}{2\sqrt{2}} \frac{y}{\sqrt{x^2+y^2}} \frac{1}{\sqrt{\sqrt{x^2+y^2}-x}} = \frac{y}{|y|} \frac{\beta(x+iy)}{2\sqrt{2}} \frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}}$$

and we must conclude that $|y|\alpha(x + iy) = y\beta(x + iy)$.

c. Let us suppose f is analytic. C can be parametrized by a continuous path γ. Then $(\mathrm{Re} f) \circ \gamma$ is continuous so if α changes sign on $C \setminus \{-R\}$ then by the intermediate value theorem there should be a point z on $C \setminus \{-R\}$ such that $\mathrm{Re} f(z) = 0$. Since this is not the case we must conclude that α is constant on $C \setminus \{-R\}$. Analogously β is constant on $C \setminus \{R\}$. This is impossible because the result of b. implies that α and β should have the same sign on the part of the circle where $\mathrm{Im}(z) > 0$ and opposite sign on the part of the circle where $\mathrm{Im}(z) < 0$. We conclude that f is not analytic.

Exercise 4. Denote by C_R the circle with radius R centered at the origin and let $f(z) = e^z$. We can parametrize the circle by Re^{it} with $0 \le t \le 2\pi$. We notice using Thm 7.3 that

$$\begin{aligned} \int_0^{2\pi} e^{R\cos(t)} \cos(R\sin(t) - nt)\, dt &= R^n \mathrm{Re} \left(\int_0^{2\pi} \frac{e^{Re^{it}}}{(Re^{it})^{n+1}} Re^{it} dt \right) \\ &= R^n \mathrm{Re} \left(\frac{1}{i} \oint_{C_R} \frac{f(z)}{z^{n+1}} dz \right) = R^n \mathrm{Re} \left(\frac{2\pi f^{(n)}(0)}{n!} \right) = \frac{2\pi R^n}{n!}. \end{aligned}$$

Exercise 5. Let $z_0 \in D(0,\rho)$. Because $D(0,\rho)$ is open we can find an $r > 0$ such that $|z_0| + r < \rho$. Now write $z = z_0 + (z - z_0)$. Then by the binomial formula we have

$$z^n = (z_0 + (z - z_0))^n = \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z - z_0)^k.$$

Now if $|z - z_0| < r$ we have $|z_0| + |z - z_0| < \rho$. Thus

$$\sum_{n=0}^{\infty} |a_n| (|z_0| + |z - z_0|)^n = \sum_{n=0}^{\infty} |a_n| \left(\sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k} |z - z_0|^k \right)$$

converges. Since the convergence is absolute we may rearrange the terms to conclude that

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right) (z - z_0)^k$$

converges absolutely for $|z - z_0| < r$. Thus f is analytic in z_0. Since z_0 was arbitrary we conclude that f is analytic on $D(0,\rho)$.