MIDTERM COMPLEX FUNCTIONS

APRIL 17 2013, 9:00-12:00

- Put your name and studentnummer on every sheet you hand in.
- When you use a theorem, show that the conditions are met.

Exercise 1 (*7 pt*) Prove that a triangle with vertices $a, b, c \in \mathbb{C}$ taken in the counter-clockwise order is equilateral if and only if

$$a + \omega b + \omega^2 c = 0,$$

where $\omega = \mathrm{e}^{i\frac{2\pi}{3}}$.

Exercise 2 (*10 pt*) Is there an analytic function $f : U \to \mathbb{C}$ defined on some open subset $U \subset \mathbb{C}$ such that

a. Re $f(z) = |z|^2$? **b**. Re $f(z) = \log(|z|^2)$?

Exercise 3 (*10 pt*) Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree $n \geq 1$ with coeffcients $a_j \in \mathbb{C}$ for $j = 0, 1, \ldots, n-1$. Prove that

$$\max_{|z|\leq 1} |P(z)| \geq 1$$

with equality attained only for $P(z) = z^n$. *Hint*: Apply the Maximun Modulus Principle for the polynomial $Q(w) = w^n P\left(\frac{1}{w}\right)$.

Exercise 4 (*8 pt*) Compute

$$\int_\gamma \left(\frac{z^2+1}{z^2-1}\right)^3 dz,$$

where γ is the circle $|z-1| = 1$ oriented counter-clockwise and traced once.

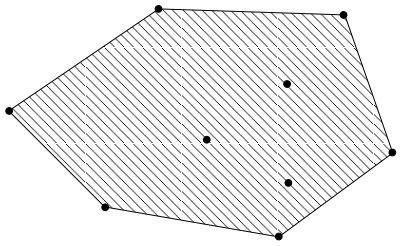
Turn the page!

Exercise 5 (*10 pt*) Is there an analytic function f on the open unit disc such that

$$f\left(\frac{i^n}{n}\right) = -\frac{1}{n^2}$$

for all $n \geq 2$?

Bonus Exercise (*10 pt*) A *convex hull* of a finite number of points $z_1, z_2, \ldots, z_n \in \mathbb{C}$ is the minimal convex subset of $\mathbb{C}$ containing all these points.



Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = \prod_{k=1}^{n}(z - z_k)$$

be a polynomial of degree $n \geq 2$ with coeffcients $a_j \in \mathbb{C}$ for $j = 0, 1, \ldots, n-1$. Prove that roots of $P'(z)$ lie in the convex hull of the roots $z_1, z_2, \ldots, z_n$ of $P(z)$ in $\mathbb{C}$.

Hint: A point $z \in \mathbb{C}$ is in the convex hull of the points $z_1, z_2, \ldots, z_n$ if and only if

$$z = \sum_{k=1}^{n} \lambda_k z_k$$

for some $\lambda_k \geq 0$ with $\displaystyle\sum_{k=1}^{n} \lambda_k = 1$.