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SOLUTIONS MIDTERM COMPLEX FUNCTIONS

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Exercise 1 ***(7 pt)*** Prove that a triangle with vertices $a, b, c \in \mathbb{C}$ taken in the counter-clockwise order is equilateral if and only if

$$a + \omega b + \omega^2 c = 0,$$

where $\omega = \mathrm{e}^{i\frac{2\pi}{3}}$.

There are several solutions possible.

1. It holds that $1 + \omega + \omega^2 = 0$. Indeed $\omega \neq 1$ but $\omega^3 = 1$, so that
$$0 = \omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1).$$
A triangle abc is equilateral if and only if its side $(c - b)$ is the side $(b - a)$ rotated through $\frac{2\pi}{3}$ counter-clockwise:

   This is the case if and only if
$$\begin{aligned} c - b = \omega(b - a) \quad &\Longleftrightarrow \quad \omega a + (-1 - \omega)b + c = 0 \\ &\Longleftrightarrow \quad \omega a + \omega^2 b + c = 0 \\ &\Longleftrightarrow \quad \omega^3 a + \omega^4 b + \omega^2 c = 0 \\ &\Longleftrightarrow \quad a + \omega b + \omega^2 c = 0, \end{aligned}$$
where it is also used that $\omega \neq 0$.
2. We notice that the equation $a+\omega b+\omega^2 c = 0$ is invariant under translation, rotation and contraction, i.e. under the transformations $(a, b, c) \to (ea + d, eb + d, ec + d)$ for $d, e \in \mathbb{C}$, for this one uses that $1 + \omega + \omega^2 = (\omega^3 - 1)(\omega - 1)^{-1} = 0$. Let us call this 'the invariance property'.
Suppose that abc is equilateral. By the invariance property we may assume that $a = 1, b = \omega$ and $c = \omega^2$. Hence $a + \omega b + \omega^2 c = 1 + \omega^2 + \omega^4 = 1 + \omega + \omega^2 = 0$.
Now suppose $a + \omega b + \omega^2 c = 0$. By the invariance property we may assume that $c = 0$. Then $a = -\omega b$, thus $|a| = |b|$. Also we find $|b-a| = |(1+\omega)b| = |-\omega^2 b| = |b|$. We conclude that abc is equilateral.

Exercise 2 ***(10 pt)*** Is there an analytic function $f : U \to \mathbb{C}$ defined on some open subset $U \subset \mathbb{C}$ such that

a. Re $f(z) = |z|^2$? **b**. Re $f(z) = \log(|z|^2)$?

a. Suppose such an f exists. Write $f = u + iv$ and $z = x + iy$, where $u = \text{Re } f$ and $v = \text{Im } f$ are C^∞ functions of (x, y). By the Cauchy-Riemann equations we must have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These equations imply that u must satisfy

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in its definition domain U. Clearly, for $u(x, y) = |z|^2 = x^2 + y^2$, we have $\Delta u = 4$ for all $(x, y) \in U$. We conclude that a function f with the desired properties does not exist.

b. Such an f exists. Let U be the complex plane minus the non-negative real numbers, and let $f(z) = 2(\log |z| + i\arg(z))$. It is known that this definition makes f analytic, and indeed $\text{Re} f(z) = \log(|z|^2)$.

Exercise 3 (*10 pt*) Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree $n \geq 1$ with coeffcients $a_j \in \mathbb{C}$ for $j = 0, 1, \ldots, n-1$. Prove that

$$\max_{|z| \leq 1} |P(z)| \geq 1$$

with equality attained only for $P(z) = z^n$.

Hint: Apply the Maximun Modulus Principle for the polynomial $Q(w) = w^n P\left(\frac{1}{w}\right)$.

Define the polynomial $Q(w) = 1 + a_{n-1}w + \ldots + a_1 w^{n-1} + a_0 w^n$. By the maximum modulus principle $|Q|$ attains its maximum on the closed unit disc in a point w with $|w| = 1$. Hence $|P(w^{-1})| = |w^n P(w^{-1})| = |Q(w)| \geq |Q(0)| = 1$. Equality is obtained precisely when the maximum of $|Q|$ is 1, i.e. when it attains its maximum in $z = 0$. The maximum modulus principle then implies that Q is constant, thus $p(z) = z^n$.

Exercise 4 (*8 pt*) Compute

$$\int_\gamma \left(\frac{z^2+1}{z^2-1}\right)^3 dz,$$

where γ is the circle $|z-1|=1$ oriented counter-clockwise and traced once.

Let

$$f(z) = \left(\frac{z^2+1}{z+1}\right)^3 = \left(z-1+\frac{2}{z+1}\right)^3.$$

By the generalized Cauchy formula we get

$$\begin{aligned}\oint_\gamma \left(\frac{z^2+1}{z^2-1}\right)^3 dz &= \oint_\gamma \frac{f(z)}{(z-1)^3} dz = \frac{2\pi i}{2!} f^{(2)}(1) \\ &= \pi i \left.\frac{d}{dz}\right|_{z=1} 3\left(1-\frac{2}{(z+1)^2}\right)\left(z-1+\frac{2}{z+1}\right)^2 \\ &= 3\pi i\left(\frac{4}{2^3}\cdot 1^2 + 2\left(1-\frac{2}{2^2}\right)^2 \cdot 1\right) = 3\pi i.\end{aligned}$$

Exercise 5 (*10 pt*) Is there an analytic function f on the open unit disc such that

$$f\left(\frac{i^n}{n}\right) = -\frac{1}{n^2}$$

for all $n \geq 2$?

Suppose that such function f exists. We notice that for the sequence $i^{2n}/(2n)$ we have $f(z) = -z^2$. Since this sequence defines a set with 0 as accumulation point we conclude that $f(z) = z^2$. But then

$$f\left(\frac{i^3}{3}\right) = \frac{1}{9} \neq -\frac{1}{3^2}.$$

Contradiction. We conclude that a function f with the desired properties does not exist.

Bonus Exercise (*10 pt*) A *convex hull* of a finite number of points $z_1, z_2, \ldots, z_n \in \mathbb{C}$ is the minimal convex subset of $\mathbb{C}$ containing all these points.

Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = \prod_{k=1}^{n}(z-z_k)$$

be a polynomial of degree $n \geq 2$ with coeffcients $a_j \in \mathbb{C}$ for $j = 0, 1, \ldots, n-1$. Prove that roots of $P'(z)$ lie in the convex hull of the roots $z_1, z_2, \ldots, z_n$ of $P(z)$ in $\mathbb{C}$.

Hint: A point $z \in \mathbb{C}$ is in the convex hull of the points $z_1, z_2, \ldots, z_n$ if and only if

$$z = \sum_{k=1}^{n} \lambda_k z_k$$

for some $\lambda_k \geq 0$ with $\displaystyle\sum_{k=1}^{n} \lambda_k = 1$.

Let w be a root of $P'(z)$. If the multiplicity of w is bigger then 1, then w is also a root of P and there is nothing left to prove. So let us suppose that w has multiplicity 1. Then we see

$$\begin{aligned} 0 = \frac{P'(w)}{P(w)} &= \frac{d}{dz}\Big|_{z=w} \log P(z) = \frac{d}{dz}\Big|_{z=w} \sum_{k=1}^{n} \log(z - z_k) \\ &= \sum_{k=1}^{n} \frac{1}{w - z_k} = \sum_{k=1}^{n} \frac{\overline{w} - \overline{z_k}}{|w - z_k|^2}. \end{aligned}$$

Taking the complex conjugate yields

$$0 = \sum_{k=1}^{n} \frac{w - z_k}{|w - z_k|^2} \text{ and thus } w = \sum_{k=1}^{n} \frac{z_k}{|w - z_k|^2 \sum_{l=1}^{n} \frac{1}{|w - z_l|^2}}$$

and we are done.