SOLUTIONS MIDTERM COMPLEX FUNCTIONS

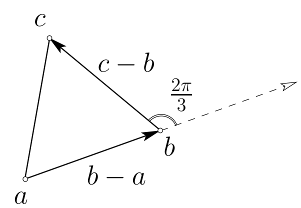
APRIL 17 2013, 9:00-12:00

Exercise 1. There are several solutions possible.

1. It holds that $1 + \omega + \omega^2 = 0$. Indeed $\omega \neq 1$ but $\omega^3 = 1$, so that

$$0 = \omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1).$$

A triangle abc is equilateral if and only if its side $(c - b)$ is the side $(b - a)$ rotated through $\frac{2\pi}{3}$ counter-clockwise:



This is the case if and only if

$$\begin{aligned} c - b = \omega(b - a) &\iff \omega a + (-1 - \omega)b + c = 0 \\ &\iff \omega a + \omega^2 b + c = 0 \\ &\iff \omega^3 a + \omega^4 b + \omega^2 c = 0 \\ &\iff a + \omega b + \omega^2 c = 0, \end{aligned}$$

where it is also used that $\omega \neq 0$.

2. We notice that the equation $a + \omega b + \omega^2 c = 0$ is invariant under translation, rotation and contraction, i.e. under the transformations $(a, b, c) \to (ea + d, eb + d, ec + d)$ for $d, e \in \mathbb{C}$, for this one uses that $1 + \omega + \omega^2 = (\omega^3 - 1)(\omega - 1)^{-1} = 0$. Let us call this 'the invariance property'.

Suppose that abc is equilateral. By the invariance property we may assume that $a = 1, b = \omega$ and $c = \omega^2$. Hence $a + \omega b + \omega^2 c = 1 + \omega^2 + \omega^4 = 1 + \omega + \omega^2 = 0$.

Now suppose $a + \omega b + \omega^2 c = 0$. By the invariance property we may assume that $c = 0$. Then $a = -\omega b$, thus $|a| = |b|$. Also we find $|b - a| = |(1+\omega)b| = |-\omega^2 b| = |b|$. We conclude that abc is equilateral.

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Exercise 2.

a. Suppose such an f exists. Write $f = u + iv$ and $z = x + iy$, where $u = \text{Re } f$ and $v = \text{Im } f$ are C^∞ functions of (x, y). By the Cauchy-Riemann equations we must have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These equations imply that u must satisfy

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in its definition domain U. Clearly, for $u(x, y) = |z|^2 = x^2 + y^2$, we have $\Delta u = 4$ for all $(x, y) \in U$. We conclude that a function f with the desired properties does not exist.

b. Such an f exists. Let U be the complex plane minus the non-negative real numbers, and let $f(z) = 2(\log |z| + i\arg(z))$. It is known that this definition makes f analytic, and indeed $\text{Re} f(z) = \log(|z|^2)$.

Exercise 3. Define the polynomial $Q(z) = 1 + a_{n-1}z + \ldots + a_1 z^{n-1} + a_0 z^n$. By the maximum modulus principle $|Q|$ attains its maximum on the closed unit disc in a point w with $|w| = 1$. Hence $|P(w^{-1})| = |w^n P(w^{-1})| = |Q(w)| \geq |Q(0)| = 1$. Equality is obtained precisely when the maximum of $|Q|$ is 1, i.e. when it attains its maximum in $z = 0$. The maximum modulus principle then implies that Q is constant, thus $p(z) = z^n$.

Exercise 4. Let

$$f(z) = \left(\frac{z^2+1}{z+1}\right)^3 = \left(z - 1 + \frac{2}{z+1}\right)^3.$$

By the generalized Cauchy formula we get

$$\begin{aligned}\oint_\gamma \left(\frac{z^2+1}{z^2-1}\right)^3 dz = \oint_\gamma \frac{f(z)}{(z-1)^3} dz &= \frac{2\pi i}{2!} f^{(2)}(1) \\ &= \pi i \left.\frac{d}{dz}\right|_{z=1} 3\left(1 - \frac{2}{(z+1)^2}\right)\left(z - 1 + \frac{2}{z+1}\right)^2 \\ &= 3\pi i \left(\frac{4}{2^3} \cdot 1^2 + 2\left(1 - \frac{2}{2^2}\right)^2 \cdot 1\right) = 3\pi i.\end{aligned}$$

Exercise 5. Since f is analytic on the open unit disc, we can represent it with a powerseries in 0. We notice that for the sequence $i^{2n}/(2n)$ we have $f(z) = -z^2$. Since this sequence defines a set with 0 as accumulation point we conclude that $f(z) = z^2$. But then

$$f\left(\frac{i^3}{3}\right) = \frac{1}{9} \neq -\frac{1}{3^2}.$$

We conclude that a function f with the desired properties does not exist.

Bonus exercise. Let w be a root of $P'(z)$. If the multiplicity of w is bigger then 1, then w is also a root of P and there is nothing left to prove. So let us suppose that w has multiplicity 1. Then we see

$$\begin{aligned} 0 &= \frac{P'(w)}{P(w)} = \frac{d}{dz}\bigg|_{z=w} \log P(z) = \frac{d}{dz}\bigg|_{z=w} \sum_{k=1}^{n} \log(z - z_k) \\ &= \sum_{k=1}^{n} \frac{1}{w - z_k} = \sum_{k=1}^{n} \frac{\overline{w} - \overline{z_k}}{|w - z_k|^2}. \end{aligned}$$

Taking the complex conjugate yields

$$0 = \sum_{k=1}^{n} \frac{w - z_k}{|w - z_k|^2} \text{ and thus } w = \sum_{k=1}^{n} \frac{z_k}{|w - z_k|^2 \sum_{l=1}^{n} \frac{1}{|w - z_l|^2}}$$

and we are done.