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SOLUTIONS ENDTERM COMPLEX FUNCTIONS

JUNE 26 2013, 9:00-12:00

Exercise 1 (*10 pt*) Give an analytic isomorphism between the first quadrant

$$Q = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \ \text{ and } \ \mathrm{Im}(z) > 0\}$$

and the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

We will construct the required analytic isomorphisn as a composition of two maps, $g : Q \to H$ and $f : H \to D$ where H denotes the upper halfplane.
Consider the function $g : Q \to H$ given by $g(z) = z^2$. Writing z in polar form, one indeed notices that g maps to H, since the argument is doubled by the mapping. We notice that an analytic inverse of g can be given by $z \mapsto \sqrt{|z|}e^{\frac{1}{2}i\log z}$ (leave out the ray of non-negative real numbers). We conclude that g is an analytic isomorphism.
Let us look for a linear fractional transformation $f : H \to D$. Such a transformation can, for example, map $\{0, i, \infty\}$ onto $\{-1, 0, 1\}$ (in this order). This defines f uniquely, namely:

$$f(z) = \frac{z - i}{z + i}.$$

Let us prove that f indeed maps to D. Write $z = x + iy$, with $y > 0$. Indeed, we have

$$\left|\frac{z - i}{z + i}\right|^2 = \frac{x^2 + (y-1)^2}{x^2 + (y+1)^2} < 1.$$

We notice that for $w \in D$

$$f^{-1}(w) = i\frac{1 + w}{1 - w}$$

and can therefore conclude that f is an analytic isomorphism. Since the composition of analytic isomorphisms is again an analytic isomorphism we conclude that $f \circ g$ is an analytic isomorphism between Q and D.

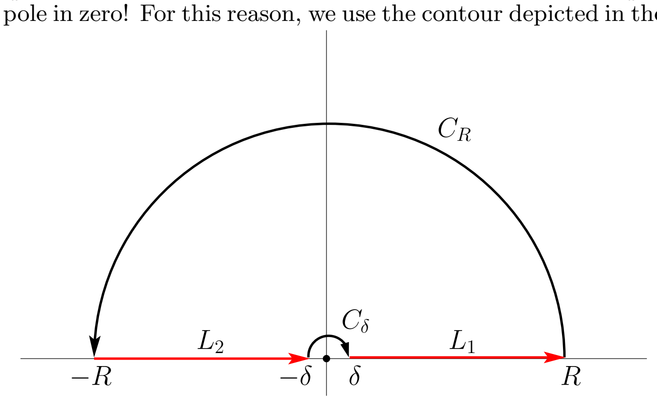
Exercise 2 (*25 pt*) Let $a, b > 0$. Prove that the following integrals converge and evaluate them.

a. (*10 pt*) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2}\, dx$

Convergence of $\int_{-\infty}^{\infty} \frac{\cos(ax)-\cos(bx)}{x^2} dx$ means that $\lim_{\substack{A \to -\infty \\ B \to \infty}} \int_A^B \frac{\cos(ax)-\cos(bx)}{x^2} dx$ exists. Since $\frac{\cos(ax)-\cos(bx)}{x^2}$ is an even function:

$$\begin{aligned}
\lim_{\substack{A \to -\infty \\ B \to \infty}} \int_A^B \frac{\cos(ax)-\cos(bx)}{x^2} dx &= \lim_{\substack{A \to \infty \\ B \to \infty}} \Big[\int_0^A \frac{\cos(ax)-\cos(bx)}{x^2} dx \\
&+ \int_0^B \frac{\cos(ax)-\cos(bx)}{x^2} dx \Big] \\
&= 2 \lim_{B \to \infty} \int_0^B \frac{\cos(ax)-\cos(bx)}{x^2} dx \\
&= \lim_{B \to \infty} \int_{-B}^B \frac{\cos(ax)-\cos(bx)}{x^2} dx
\end{aligned}$$

We'll show that the last limit exists. First, notice that $\cos(ax) - \cos(bx) = \mathrm{Re}(e^{iax} - e^{bxi})$ and therefore it suffices to compute $\lim_{B\to\infty} \mathrm{Re}(\int_{-B}^B \frac{e^{iax}-e^{bxi}}{x^2} dx)$. With l'Hôpital, or the fact that $\cos(ax) - \cos(bx) = x^2(a^2 - b^2)/2 + O(x^4)$, we can see that $\frac{\cos(ax)-\cos(bx)}{x^2}$ has no (non-removable) singularity on $\mathbb{R}$, but $\frac{e^{iax}-e^{bxi}}{x^2}$ does have a simple pole in zero! For this reason, we use the contour depicted in the image.



Of course, we take $\delta \downarrow 0$ and $R \to \infty$. The lemma on page 196 of Lang implies that $\lim_{\delta\downarrow 0} \int_{C_\delta} \frac{e^{iax}-e^{bxi}}{x^2} dx = -\pi i \mathrm{Res}_{x=0}(\frac{e^{iax}-e^{bxi}}{x^2}) = \pi(a-b)$, where the minus sign comes from the fact that our path is clockwise orientated. (Cauchy's theorem, which is used in the prove of the lemma, works for counter-clockwise orientated). Showing $\lim_{R\to\infty} \int_{C_R} \frac{e^{iax}-e^{bxi}}{x^2} dx = 0$ is straight forward.
Notice that there are no poles inside our contour and therefore:

$$\lim_{R\to\infty, \delta\downarrow 0} \Big[\int_{C_\delta} + \int_{L_1} + \int_{C_R} + \int_{L_2} \Big] = \int_{\mathbb{R}} + \pi(a-b) = 0$$

Clearly, our integral equals $\pi(b-a)$.

Remark. For the convergence of the integral one can also use that $(\cos(ax) - \cos(bx))/x^2 \leq 2/x^2$ and thus it converges by the comparison test.

b. *(15 pt)* $\displaystyle\int_{-\infty}^{\infty} e^{-ax^2}\cos(bx)\,dx$ (*Hint*: Use a rectangular countour.)

Convergence follows in the same way as above when we remember $\int_{\mathbb{R}} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$. As usual, we write $\cos(bx) = \mathrm{Re}(e^{ibx})$ such that we get $\int_{\mathbb{R}} e^{-ax^2+ibx}$. The idea is to transform our integral to a gaussian integral as above and we will achieve this with a translation of a complex number t: $x \mapsto x + t$. We get

$$-ax^2 + ibx \mapsto -a(x+t)^2 + ib(x+t) = -ax^2 + (ib - 2at)x + (-at^2 + ibt)$$

Notice that taking $t = i\frac{b}{2a}$ ensures that the linear term on the right vanishes, it equals $-ax^2 + (-at^2 + ibt)$. For this reason, let the contour be the rectangle formed by $[-R, R, R + i\frac{b}{2a}, -R + i\frac{b}{2a}]$, orientated counter-clockwise. Since $f(x) = e^{-ax^2+ibx}$ is an entire function, the integral will equal zero. It is easy to prove that the vertical path's will not contribute, i.e. $\lim_{R\to\infty}\int_R^{R+i\frac{b}{2a}} f(x)dx = \lim_{R\to\infty}\int_{-R+i\frac{b}{2a}}^{-R} f(x)dx = 0$. We end up with:

$$\int_{-R}^{R} e^{-ax^2+ibx}dx = -\int_{R+i\frac{b}{2a}}^{-R+i\frac{b}{2a}} e^{-ax^2+ibx}dx = \int_{-R+i\frac{b}{2a}}^{R+i\frac{b}{2a}} e^{-ax^2+ibx}dx$$

Notice that, because of the smart choice of $t = i\frac{b}{2a}$, letting $R \to \infty$, the last integral is simply $\int_{-\infty}^{\infty} e^{-ax^2+(-at^2+ibt)}dx = e^{-at^2+ibt}\sqrt{\frac{\pi}{a}}$. Taking the real part, we can conclude $\int_{-\infty}^{\infty} e^{-ax^2}\cos(bx)dx = e^{b^2/(2a)}\sqrt{\frac{\pi}{a}}$.

Remark. For the convergence of the integral one can also use that

$$|e^{-ax^2}\cos(bx)| \leq e^{-ax^2}$$

and thus it converges by the comparison test.

Exercise 3 (*10 pt*) Consider the polynomial function $P(z) = z^7 - 2z - 5$.

a. *(7 pt)* Determine the number of roots of P with $\mathrm{Re}(z) > 0$.

We consider the contour γ_R given by ($R > 0$):

$$L_R(t) = it \text{ with } t \in [-R, R]$$
$$C_R(t) = Re^{it} \text{ with } t \in [-\pi/2, \pi/2]$$

We define $Q(z) = z^7 - 5$. In order to apply Rouché's theorem we will prove that $|P(z) - Q(z)| < |Q(z)|$ on γ_R. On C_R we can obviously pick R big enough to achieve this. On L_R, let us first consider the case that $|t| < 5/2$. Then we see that

$$|P(it) - Q(it)| = 2|t| < 5 < |-it^7 + 5| = |Q(it)|.$$

For $|t| \geq 5/2$, we have

$$|P(it) - Q(it)|^2 - |Q(it)|^2 = 4t^2 - (t^{14} + 25) = -t^2(t^{12} - 4) - 25 < 0.$$

It remains to show that P does not have a root on γ_R. Since it has only finitely many roots we can pick R big enough such that none of its roots are on C_R. There is also no root on L_R, since

$$|P(it)|^2 = t^2(t^6+2)^2 + 25 > 0.$$

Thus we may apply Rouché's theorem to conclude that P has just as many roots inside γ_R as Q. Since Q has the roots $5^{1/7}, 5^{1/7}e^{2\pi i/7}, 5^{1/7}e^{-2\pi i/7}$ we conclude that P has three roots (counted with multiplicity) in the region with $\mathrm{Re}(z) > 0$.

Alternative solution. We know that

$$\frac{1}{2\pi i}\int_{\gamma_R} \frac{P'(z)}{P(z)} dz = \text{number of roots of } P \text{ (counted with multiplicity).}$$

By log we denote the logarithm with argument in $(0, 2\pi)$. We notice that

$$\int_{L_R} \frac{P'(z)}{P(z)} dz = [\log P(z)]_{iR}^{-iR} = \log(-5 + iR(R^6+1)) - \log(-5 - iR(R^6+1))$$
$$\to \frac{\pi i}{2} - \frac{3\pi i}{2} = -\pi i \text{ as } R \to \infty.$$

We notice that

$$\int_{C_R} \frac{P'(z)}{P(z)} dz = i\int_{-\pi/2}^{\pi/2} \frac{7R^7e^{7it} - 2Re^{it}}{R^7e^{7it} - 2R^2e^{2it} - 5} dt \to i\int_{-\pi/2}^{\pi/2} 7dt = 7\pi i$$

as $R \to \infty$, and we are done.

b. *(3 pt)* How many of them are simple?

Suppose P has a root w (with $\mathrm{Re}(w) > 0$) of multiplicity > 1. Then we have $0 = P'(w) = 7w^6 - 2$, thus $w = (\frac{2}{7})^{1/6}e^{2\pi ik/7}$ for some $k \in \{-1, 0, 1\}$. However, then we would have

$$|w^7 - 2w - 5| = \left|\frac{-12}{7}w - 5\right| \geq 5 - \frac{12}{7}\left(\frac{2}{7}\right)^{1/6} > 0.$$

We conclude that the three roots are simple.

Bonus Exercise *(15 pt)* Prove that

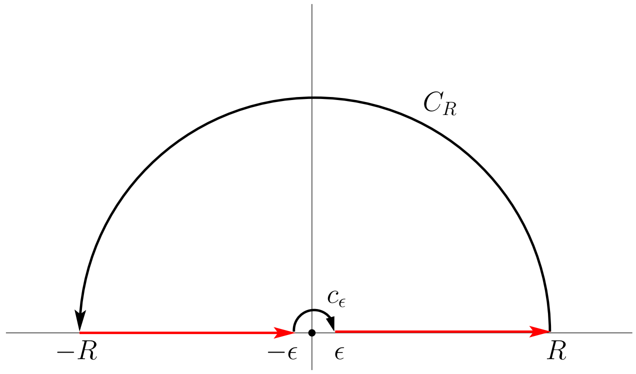
$$\int_0^\infty \frac{\sin(x)}{\log^2(x) + \frac{\pi^2}{4}}\, dx = \frac{2}{e} + \frac{2}{\pi}\int_0^\infty \frac{\log(x)\cos(x)}{\log^2(x) + \frac{\pi^2}{4}}\, dx\ .$$

You may assume that the integrals converge.

Let us define the function $f : \mathbb{C} \setminus \{iy | y \leq 0\} \to \mathbb{C}$ by

$$f(z) = \frac{e^{iz}}{\log(z) - \pi i/2}$$

where the argument is chosen in $(-\pi/2, 3\pi/2)$. Let us integrate f over the following contour.



For the integral over the little semicircle c_ϵ we have

$$\lim_{\epsilon\downarrow 0}\left|\int_{c_\epsilon} f\right| \leq \lim_{\epsilon\downarrow 0}\int_0^\pi \frac{e^{-\epsilon\sin(t)}}{|\log(\epsilon)+i(t-\pi i/2)|}\epsilon dt \leq \lim_{\epsilon\downarrow 0}\frac{\pi\epsilon}{-\log\epsilon} = 0.$$

For the integral over the big semicircle C_R we will use the inequality $\sin(t) \geq \frac{\pi}{2}t$ for $0 \leq t \leq \frac{\pi}{2}$. We see

$$\left|\int_{C_R} f\right| \leq \int_0^\pi \frac{e^{-R\sin(t)}R}{|\log(R)+i(t-\pi i/2)|}dt = 2\int_0^{\frac{\pi}{2}} \frac{e^{-R\sin(t)}R}{|\log(R)+i(t-\pi i/2)|}dt$$
$$\leq 2\int_0^{\frac{\pi}{2}} \frac{e^{-\frac{\pi}{2}Rt}R}{\log R}dt = \frac{4}{\pi}\frac{1-e^{-R}}{\log R} \to 0 \text{ as } R\to\infty.$$

By the residue theorem we get

$$-\int_0^\infty \frac{e^{-it}}{\log(t)+\pi i-\pi i/2}(-1)dt + \int_0^\infty \frac{e^{it}}{\log(t)-\pi i/2}dt = 2\pi i \lim_{z\to i}\frac{z-i}{\log(z)-\log(i)}e^{iz} = -\frac{2\pi}{e}.$$

Working this out leads to the required equality.