Docent: Yu.A. Kuznetsov
Assistants: L. Molag, W. Pranger, B. Nieraeth

SOLUTIONS ENDTERM COMPLEX FUNCTIONS

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Exercise 1 (*10 pt*): Consider a transformation of the complex plane

$$z \mapsto a\bar{z} + b,$$

where $a, b \in \mathbb{C}$ with $|a| = 1$. Prove that this transformation has a straight line composed of fixed points if and only if

$$-a\bar{b} = b.$$

Let f denote the transformation $z \mapsto a\bar{z} + b$.

($\Rightarrow$) Assume that there is a line in the complex plane composed of fixed points of f, i.e. $f(z) = a\bar{z} + b = z$. It should also hold that $f \circ f(z) = f(f(z)) = f(z) = z$. If we write out what this means, we get the following: $f(f(z)) = f(a\bar{z}+b) = a(\bar{a}z+\bar{b})+b = z+a\bar{b}+b = z$. Here we used $|a| = 1$ and we conclude that $a\bar{b} = -b$.

($\Leftarrow$) Assume that $a\bar{b} = -b$. Plugging $\frac{b}{2}$ into f, we get $f(\frac{b}{2}) = a(\overline{\frac{b}{2}})+b = -\frac{b}{2}+b = \frac{b}{2}$ i.e. $\frac{b}{2}$ is a fixed point of f. A line in $\mathbb{C}$ which goes through this point is of the form $\frac{b}{2} + Re^{it}$ for $R \in \mathbb{R}$ and some real value t which we need find. Notice that $|a| = 1$ implies that $a = e^{i\phi}$ for some real value ϕ.

$$f(\frac{b}{2} + Re^{it}) = e^{i\phi}(\overline{\frac{b}{2} + Re^{it}}) + b = Re^{i\phi - it} + \frac{b}{2}$$

Solving $f(\frac{b}{2} + Re^{it}) = \frac{b}{2} + Re^{it}$ for t yiels $t = \frac{\phi}{2}$ and it follows that $R \mapsto \frac{b}{2} + Re^{i\frac{\phi}{2}}$ is a line composed of fixed points of f.

Exercise 2 (*10 pt*): Let $m > 0$ be integer. Find the convergence radius of the following series

$$\sum_{n=0}^{\infty}(a_1^n + a_2^n + \cdots + a_m^n)z^n,$$

where $a_j \in \mathbb{C}$ with $|a_j| = 1$ for $j = 1, 2, \ldots, m$.

Note that

$$f : z \mapsto \sum_{j=1}^{m} \frac{1}{1 - a_j z} \tag{1}$$

is analytic in $\mathbb{C}\backslash\{\overline{a_1},\ldots,\overline{a_m}\}$. Since the power series expansion of f about 0 has a radius of convergence that is "as big as possible" (see the last assertion of Theorem 7.3 on page 128 of Lang), we find that its radius of convergence is at least 1. Since none of the terms in the sum in (1) cancel each other out we find that the radius of convergence must be equal to 1. Note that for $|z| < 1$ we have

$$\begin{aligned}\sum_{n=0}^{\infty}(a_1^n + \cdots + a_m^n)z^n &= \lim_{N\to\infty}\sum_{n=0}^{N}(a_1^n + \cdots + a_m^n)z^n \\ &= \lim_{N\to\infty}\sum_{n=0}^{N}(a_1 z)^n + \cdots + \lim_{N\to\infty}\sum_{n=0}^{N}(a_m z)^n \\ &= \frac{1}{1-a_1 z} + \cdots + \frac{1}{1-a_m z} = f(z).\end{aligned}$$

Hence, by Theorem 3.2 on page 62 of Lang we find that the series for f about 0 coincides with the series $\sum_{n=0}^{\infty}(a_1^n + + \cdots + a_m^n)z^n$. We conclude that the radius of convergence of our series is 1.

Exercise 3 (*15 pt*): Let $\Omega \subset \mathbb{C}$ be open and bounded. We define the *Cauchy-Riemann operator* by

$$\partial_{\overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Prove that the boundary value problem

$$\begin{cases} \partial_{\overline{z}} u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

for given continuous functions $f : \Omega \to \mathbb{C}$ and $g : \partial\Omega \to \mathbb{C}$ has at most one solution $u : \overline{\Omega} \to \mathbb{C}$ that is continuous in $\overline{\Omega}$.

Let w_1 and w_2 denote the real and imaginary part of a function $w : \Omega \to \mathbb{C}$ so that $w = w_1 + iw_2$. Note that

$$2\partial_{\overline{z}} w = \frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial y} + i\left(\frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial x}\right).$$

This shows us that w is holomorphic in Ω, or equivalently, satisfies the Cauchy-Riemann equations in Ω if and only if $\partial_{\overline{z}} w = 0$ in Ω.
Now suppose $u, v : \overline{\Omega} \to \mathbb{C}$ are both continuous on $\overline{\Omega}$ and satisfy the given equation. Their difference $w := u - v$ is then also continuous in $\overline{\Omega}$ and satisfies

$$\begin{cases} \partial_{\overline{z}} w = f - f = 0 & \text{in } \Omega \\ w|_{\partial\Omega} = g - g = 0. \end{cases}$$

By compactness of $\overline{\Omega}$ we find that $|w|$ attains its maximum on $\overline{\Omega}$. Since w is holomorphic in Ω we may apply the Maximum Modulus Principle to each connected component of Ω (cf. Corollary 1.4 on page 92 of Lang) to find that this maximum must be on $\partial\Omega$. This implies that $|u(z) - v(z)| = |w(z)| \leq 0$ for all $z \in \overline{\Omega}$. We conclude that $u = v$.

Exercise 4 (*20 pt*): Let $f(z) = z^6 - 5z^4 + 10$.

(a) (*15 pt*) Prove that f has

(i) no zeroes with $|z| < 1$;
(ii) 4 zeroes with $|z| < 2$;
(iii) 6 zeroes with $|z| < 3$.

(b) (*5 pt*) For cases (ii) and (iii), show that all zeroes are different.

(a) We will apply Rouché's Theorem 1.6 (p. 181 of Lang) with respect to three different polynomials.

(i) Set $g(z) = 10$ and let $|z| = 1$. We get the following:

$$|f(z) - g(z)| = |z^6 - 5z^4| \leq |z|^6 + 5|z|^4 = 1 + 5 < 10 = |g(z)|$$

It follows that we may apply Rouché and we conclude that $f(z)$ and $g(z)$ have the same number of roots in $|z| < 1$ i.e. $f(z)$ has no roots in $|z| < 1$.

(ii) Set $g(z) = -5z^4$ and $|z| = 2$. We get the following:

$$|f(z) - g(z)| = |z^6 + 10| \leq |z|^6 + 10 = 74 < 80 = 5 \cdot 2^4 = |g(z)|$$

It follows that we may apply Rouché and we conclude that $f(z)$ and $g(z)$ have the same number of roots in $|z| < 2$ i.e. $f(z)$ has four roots in $|z| < 2$.

(iii) Set $g(z) = z^6$ and $|z| = 3$. We get the following:

$$|f(z) - g(z)| = |-5z^4 + 10| \leq 5|z|^4 + 10 = 415 < 729 = 810 - 81 = 9 \cdot 81 = 3^6 = |g(z)|$$

It follows that we may apply Rouché and we conclude that $f(z)$ and $g(z)$ have the same number of roots in $|z| < 3$ i.e. $f(z)$ has six roots in $|z| < 3$.

(b) We need to prove that the roots of $f(z)$ have multiplicity one, i.e. want to prove that $f'(z) = 6z^5 - 20z^3$ has no roots in common with $f(z)$. Solving $f'(z) = 0$ we see that $z^3 = 0$ or $6z^2 - 20 = 0$ i.e. $z = 0$ or $z = \pm\sqrt{\frac{10}{3}}$. Since $f(0) \neq 0$ we only need to check that $f(\pm\sqrt{\frac{10}{3}}) \neq 0$.

$$f(\pm\sqrt{\frac{10}{3}}) = (\frac{10}{3})^3 - 5(\frac{10}{3})^2 + 10 = \frac{10^3 - 5 \cdot 3 \cdot 10^2 + 27 \cdot 10}{27} = -\frac{230}{27}.$$

It follows that $f(\pm\sqrt{\frac{10}{3}}) \neq 0$ and we conclude that the roots of f are simple.

Exercise 5 (*25 pt*): Prove that the integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx$$

converges and compute it.

Hint: As one possibility is to consider the integral of the function $\dfrac{e^{iz}}{z^3}$ over an appropriate closed path and prove that

$$\int_\rho^\infty \frac{\sin x}{x^3} dx = \frac{1}{\rho} - \frac{\pi}{4} + O(\rho), \quad \rho \to 0,$$

from which one can deduce

$$\int_\rho^\infty \frac{\sin 3x}{x^3} dx = \frac{3}{\rho} - \frac{9\pi}{4} + O(\rho), \quad \rho \to 0.$$

We will consider $f(z) = e^{iz}/z^3$. Let $R > 1$ and $\epsilon < 1$. We take the contour consisting of a large semicircle (in the upper halfplane) C_R from R to $-R$, a small semicircle (in the upper halfplane) c_ϵ from $-\epsilon$ to ϵ and two linesegments L_1 and L_2 from ϵ to R and from $-R$ to $-\epsilon$. In the upper half plane we have $|e^{iz}| \leq 1$, thus

$$\left| \int_{C_R} f \right| \leq \frac{1}{R^3} \pi R \to 0 \text{ as } R \to \infty.$$

For the small semicircle we will have to rewrite f in a convenient way, namely

$$f(z) = \frac{1}{z^3} + \frac{i}{z^2} - \frac{1}{2}\frac{1}{z} + \frac{e^{iz} - 1 - iz + \frac{1}{2}z^2}{z^3}$$

The term on the right extends to a continuous function, let's call it h, thus

$$\left| \int_{c_\epsilon} \frac{e^{iz} - 1 - iz + \frac{1}{2}z^2}{z^3} \right| \leq \max_{c_\epsilon} |h| \cdot \pi\epsilon \leq \max_{|z|\leq 1} |h| \cdot \pi\epsilon = \mathcal{O}(\epsilon)$$

The remaining terms are simply a matter of filling in the path ϵe^{it} from π to 0, and integrating the terms, yielding

$$\begin{aligned} \int_{c_\epsilon} \left(\frac{1}{z^3} + \frac{i}{z^2} - \frac{1}{2}\frac{1}{z} \right) dz &= \int_\pi^0 \left(i\epsilon^{-2}e^{-2it} - \epsilon^{-1}e^{-it} - \frac{1}{2}i \right) dt \\ &= \frac{1}{2}\epsilon^{-2}e^{-2it} + ie^{-it} + \frac{1}{2}i\epsilon^{-1}t \Big|_0^\pi = i\frac{\pi}{2} - 2i\epsilon^{-1}. \end{aligned}$$

Now the residue theorem (or Cauchy's theorem) implies that

$$\lim_{R\to\infty} \left(\int_{-R}^{-\epsilon} \frac{e^{ix}}{x^3} dx + \int_\epsilon^R \frac{e^{ix}}{x^3} dx \right) + i\frac{\pi}{2} - 2i\epsilon^{-1} + \mathcal{O}(\epsilon) = 0.$$

The substitution $x \to -x$ for the first integral yields

$$\int_\epsilon^\infty \frac{\sin(x)}{x^3} dx = -\frac{\pi}{4} + \frac{1}{\epsilon} + \mathcal{O}(\epsilon)$$

The substitution $x \to x/3$ yields

$$\begin{aligned} \int_\epsilon^\infty \frac{\sin(3x)}{x^3} dx &= \int_{3\epsilon}^\infty \frac{\sin(x)}{\frac{1}{27}x^3} \frac{1}{3} dx \\ &= 9\left(-\frac{\pi}{4} + \frac{1}{3\epsilon} + \mathcal{O}(3\epsilon) \right) = -\frac{9\pi}{4} + \frac{3}{\epsilon} + \mathcal{O}(\epsilon). \end{aligned}$$

We have the identity $(2i)^3 \sin^3(x) = e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix} = 2i\sin(3x) - 6i\sin(x)$, thus

$$\int_\epsilon^\infty \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{8} + \mathcal{O}(\epsilon).$$

The limit $\epsilon \to 0$ clearly exists, which proves the convergence (another argument for this is that the integrand is $\mathcal{O}(x^{-3})$ as $|x| \to \infty$ and can be continuously extended around $x = 0$). Using the fact that the integrand is even gives

$$\int_{-\infty}^\infty \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{4},$$

Bonus Exercise (*10 pt*): Let a function $f : \mathbb{C} \to \mathbb{C}$ be continuous. Suppose moreover that f is analytic for both Re $z > 0$ and Re $z < 0$. Prove that f is analytic on $\mathbb{C}$.

It suffices to prove that f is analytic in $z = 0$ (by translation). Now let us define

$$g(z) = \sum_{n=0}^\infty a_n z^n \text{ with } a_n = \frac{1}{2\pi i}\oint_{|z|=1} \frac{f(w)}{w^{n+1}} dw.$$

This definition feels natural: g should equal f, provided that f really is analytic. The expression is being used in the proof of the theorem about removable singularities (p. 165), so this seems a fruitful approach. The continuity of f ensures that the integrals defining the a_n are well-defined, and also that f attains a maximum on every compact set. Hence for $|z| < 1$

$$\sum_{n=0}^\infty |a_n z^n| \le \frac{1}{2\pi} 2\pi \max_{|w|=1} |f(w)| \sum_{n=0}^\infty |z|^n = \max_{|w|=1} |f(w)| \frac{1}{1-|z|} < \infty.$$

We conclude that g defines an analytic function on the open unit disc, furthermore by uniform convergence

$$g(z) = \frac{1}{2\pi i}\oint_{|w|=1} \frac{f(w)}{w} \sum_{n=0}^\infty \left(\frac{z}{w}\right)^n dw = \frac{1}{2\pi i}\oint_{|w|=1} \frac{f(w)}{w-z} dw$$

for $|z| < 1$. Notice that one can recognize Cauchy's integral formula in this expression, if f is indeed analytic.

Now fix a z in the open unit disc that is not purely imaginary. Let $0 < \epsilon < \frac{\pi}{2}$. We choose an $0 < \eta \le \epsilon$ such that $|\text{Re}(z)| > \eta$. Now devide the path $|z| = 1$ in four circle segments: C_1 from $-ie^{i\eta}$ to $ie^{-i\eta}$, followed by C_2 to $ie^{i\eta}$, followed by C_3 to $-ie^{-i\eta}$, followed by C_4. The line segments from $ie^{-i\eta}$ to $-ie^{i\eta}$ and $-ie^{-i\eta}$ to $ie^{i\eta}$ are referred to as L_1 and L_2 (respectively). For notational convenience define $I(w) = f(w)/(w-z)$ and define M as the maximum of $|I|$ on the unit circle. Notice that

$$2\pi i g(z) = \oint_{C_1+C_2+C_3+C_4} I = \oint_{C_1+L_1} I + \oint_{C_3+L_2} I + \int_{C_2} I + \int_{C_4} I - \int_{L_1} I - \int_{L_2} I$$

By the residue theorem the integrals over $C_1 + L_1$ and $C_3 + L_2$ (combined) contribute $2\pi i f(z)$. The integrals over C_2 and C_4 are less then or equal to $2\epsilon M$. By making η smaller (if necessary) we can arrange that

$$\max_{w \in L_1} |I(w) - I(w - 2\sin(\eta))| \leq \epsilon.$$

To see why this is true, suppose it isn't. Then for every integer $m \geq 1/\eta$ there exists a $w_m \in L_1$ such that $|I(w_m) - I(w_m - 2\sin(1/m)| \geq \epsilon$. Since L_1 is compact we find a convergent subsequence w_{m_j}. However, $|I(w_{m_j}) - I(w_{m_j} - 2\sin(1/m_j)|$ goes to 0 as $j \to \infty$ (by continuity) while it should be $\geq \epsilon$, a contradiction. We infer that

$$\left| \int_{L_1} I + \int_{L_2} I \right| = \left| \int_{L_1} (I(w) - I(w - 2\sin(\eta)))dw \right| \leq \epsilon.$$

Therefore we have proved that $|g(z) - f(z)| \leq (4M+1)\epsilon/(2\pi)$ for all $0 < \epsilon < \frac{\pi}{2}$, hence equals 0. Since z was arbitrary, we conclude that $f = g$ on an open set. Analytic continuation, combined with the fact that the continuous extension to the imaginary axis is unique, finishes the proof.

Alternative. According to the reverse of Goursat's theorem, i.e. Theorem 3.2 (p. 108), f is analytic if the integral of f over any rectangle is 0. Obviously, the integral of f along a rectangle that does not intersect the imaginary axis is 0, by Cauchy's (or residue) theorem. Suppose D is a rectangle that is divided into two rectangles by the imaginary axis. Without loss of generality D is the rectangle with vertices $1, 1+i, -1+i, -1$. Let $0 < \epsilon < 1$ and take $0 < \eta \leq \epsilon$. Now we divide D into four paths, a line segment L_2 from $\eta + i$ to $-\eta + i$, a line segment L_4 from $-\eta$ to η, and the two paths connecting them by L_1 (positive real part) and L_3. The line segments from $\eta + i$ to η and $-\eta$ to $-\eta + i$ are denoted by L_5 and L_6 respectively. We have

$$\oint_D f = \oint_{L_1 + L_5} f + \oint_{L_3 + L_6} f + \int_{L_2} f + \int_{L_4} f - \int_{L_5} f - \int_{L_6} f.$$

Denote by M the maximum of $|f|$ on D. We know that the integral of f over L_2 and L_4 are $\leq 2M\epsilon$. By Cauchy's theorem (or the residue theorem) the integrals over $L_1 + L_5$ and $L_3 + L_6$ are zero. By making η smaller (if necessary) we can arrange that (see above solution)

$$\max_{w \in L_5} |f(w) - f(w - 2\eta)| \leq \epsilon.$$

Hence

$$\left| \int_{L_5} f + \int_{L_6} f \right| = \left| \int_{L_5} (f(w) - f(w - 2\eta))dw \right| \leq \epsilon.$$

We conclude that the integral of f over D is $\leq (4M+1)\epsilon$ for all $0 < \epsilon < 1$, hence is 0. (The case where an edge of D coincides with the imaginary axis is treated in a similar fashion.)