RETAKE COMPLEX FUNCTIONS

AUGUST 20 2014, 9:00-12:00

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.

Exercise 1 (*20 pt*): Consider a transformation of the complex plane

$$z \mapsto a\bar{z} + b, \tag{1}$$

where $a, b \in \mathbb{C}$ are such that $|a| = 1$ and $a\bar{b} = -b$. It is known that this transformation has a straight line composed of fixed points. Prove that (1) acts as a mirror reflection in this line.

Exercise 2 (*10 pt*): Find the convergence radius of the series

$$\sum_{n=1}^{\infty} z^{n!} \ ,$$

where $n! = 1 \cdot 2 \cdots (n-1) \cdot n$.

Exercise 3 (*50 pt*): Let $0 < a < b$ and let

$$I := \frac{1}{\pi} \int_a^b \frac{\sqrt{(x-a)(b-x)}}{x} \, dx \ . \tag{2}$$

a. **(*10 pt*)** Prove that there exists an analytic function $g : \mathbb{C} \setminus [a,b] \to \mathbb{C}$ such that $[g(z)]^2 = (z-a)(b-z)$ for $z \in \mathbb{C} \setminus [a,b]$ while

$$\begin{aligned} \lim_{\varepsilon \uparrow 0} g(x + i\varepsilon) &= \sqrt{(x-a)(b-x)}, \\ \lim_{\varepsilon \downarrow 0} g(x + i\varepsilon) &= -\sqrt{(x-a)(b-x)} \end{aligned}$$

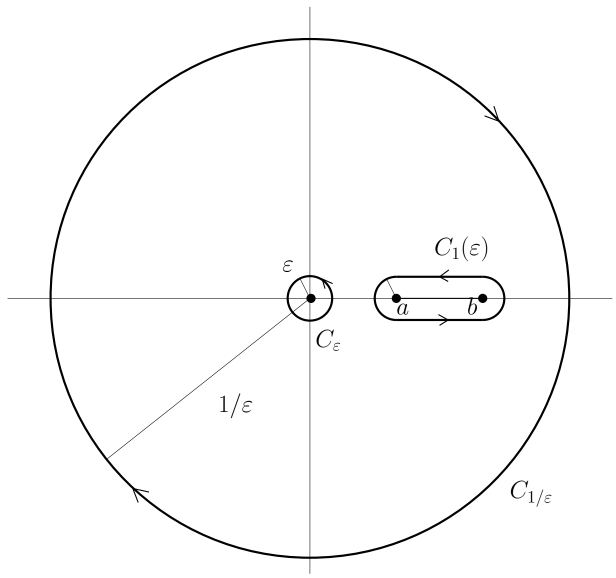
for $x \in [a,b]$.

Turn the page!

b. ***(10 pt)*** Consider the integral

$$\int_C \frac{g(z)}{z} dz$$

over the closed chain $C = C_\varepsilon + C_{1/\varepsilon} + C_1(\varepsilon)$ shown below



with ε small enough. Argue that this integral vanishes.

c. ***(10 pt)*** Evaluate the integrals over the circles

$$\int_{C_\varepsilon} \frac{g(z)}{z} dz \quad \text{and} \quad \int_{C_{1/\varepsilon}} \frac{g(z)}{z} dz$$

using residues. *Hint*: Substitute $w = 1/z$ in the second integral.

d. ***(10 pt)*** Prove that

$$\lim_{\varepsilon \downarrow 0} \int_{C_1(\varepsilon)} \frac{g(z)}{z} dz = 2\pi I,$$

where $C_1(\varepsilon)$ is the path near the segment $[a, b]$ and I is the integral (2).

e. ***(10 pt)*** Combine the obtained results to explicitly evaluate I.

Exercise 4 (*20 pt*): Let $r, R \in \mathbb{R}$ such that $0 < r < R$. Denote by D the open unit disc in $\mathbb{C}$. Prove that there is no analytic isomorphism from the punctured disc $D \setminus \{0\}$ to the annulus $A := \{z \in \mathbb{C} : r < |z| < R\}$.