ENDTERM COMPLEX FUNCTIONS

JUNE 30 2015, 9:00-12:00

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.

Exercise 1 (*10 pt*): Let α, β, γ be three different complex numbers satisfying

$$\frac{\beta - \alpha}{\gamma - \alpha} = \frac{\alpha - \gamma}{\beta - \gamma} \ .$$

Prove that the triangle with vertices $\{\alpha, \beta, \gamma\}$ is equilateral, i.e.

$$|\beta - \alpha| = |\gamma - \alpha| = |\beta - \gamma|.$$

Exercise 2 (*10 pt*): Find all entire functions f such that $|f'(z)| < |f(z)|$ for all $z \in \mathbb{C}$.

Exercise 3 (*15 pt*): Consider the polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

with real coefficients $a_k \in \mathbb{R}$ satisfying

$$a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n > 0 \ .$$

Prove that this equation has no roots with $|z| < 1$.

Exercise 4 (*20 pt*): Let f be a meromorphic function on $\mathbb{C}$. Suppose there exist $C, R > 0$ and integer $n \geq 1$ such that $|f(z)| \leq C|z|^n$ for all $z \in \mathbb{C}$ with $|z| \geq R$.

a. (*10 pt*) Prove that the number of poles of f in $\mathbb{C}$ is finite.

b. (*10 pt*) Prove that f is a rational function, i.e. it can be written as a ratio of two polynomials.

Turn the page!

Exercise 5 (*25 pt*): Let $a > 0$. By integrating the function

$$f(z) = \frac{1}{z} \, \frac{1}{\cos(2\pi i a) - \cos(2\pi z)}$$

over a suitable closed path, show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi}{a} \frac{e^{2\pi a} - e^{-2\pi a}}{e^{2\pi a} + e^{-2\pi a} - 2}.$$

Hint: Use a square path.

Bonus Exercise (*20 pt*): Find all entire functions f such that

$$f(z^2) = (f(z))^2$$

for all $z \in \mathbb{C}$.