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SOLUTIONS ENDTERM COMPLEX FUNCTIONS

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Exercise 1 (*10 pt*): Let α, β, γ be three different complex numbers satisfying

$$\frac{\beta - \alpha}{\gamma - \alpha} = \frac{\alpha - \gamma}{\beta - \gamma} \ .$$

Prove that the triangle with vertices $\{\alpha, \beta, \gamma\}$ is equilateral, i.e.

$$|\beta - \alpha| = |\gamma - \alpha| = |\beta - \gamma|.$$

Solution 1: Both the property that α, β, γ are the vertices of an equilateral triangle and the property that they satisfy

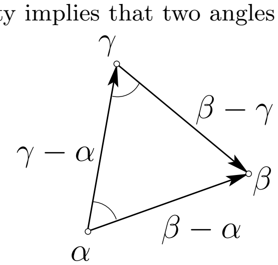
$$\frac{\beta - \alpha}{\gamma - \alpha} = \frac{\alpha - \gamma}{\beta - \gamma}$$

are invariant under translations, therefore we may take $\alpha = 0$ without loss of generality. Both properties are also invariant under rotations and rescaling (i.e. a multiplication by some complex number C). Therefore we may take $\gamma = 1$ without loss of generality. We are then left with

$$\beta = \frac{-1}{\beta - 1} \ .$$

This yields the quadratic equation $\beta^2 - \beta + 1 = 0$, which has the solutions $e^{\pi i/3}$ and $e^{-\pi i/3}$. Indeed $\{0, 1, e^{\pm \pi i/3}\}$ defines an equilateral triangle.

Solution 2: The given equality implies that two angles (at α and γ) in the triangle are equal:



To see this, use the geometric interpretation of the division. Rewriting the equality as

$$\frac{\beta - \gamma}{\alpha - \gamma} = \frac{\gamma - \alpha}{\beta - \alpha}$$

shows that the angles at γ and α are also equal. So all angles are equal, implying that the triangle is equilateral.

Exercise 2 (*10 pt*): Find all entire functions f such that $|f'(z)| < |f(z)|$ for all $z \in \mathbb{C}$.

Let f be such a function. It follows from the strict inequality that f cannot have zeros. Therefore the function f'/f is a well-defined entire function. In particular, it is bounded by 1. By Liouville's theorem this implies that f'/f is a constant function. Thus there exists a constant $c \in \mathbb{C}$ such that $f' = cf$. Then we must conclude that $f(z) = be^{cz}$, where $|c| < 1$ and $b = f(0) \in \mathbb{C}$ is arbitrary.

Exercise 3 (*15 pt*): Consider the polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

with real coefficients $a_k \in \mathbb{R}$ satisfying

$$a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n > 0 \ .$$

Prove that this equation has no roots with $|z| < 1$.

Suppose z is a root of $a_n z^n + \ldots + a_0$ with $|z| < 1$. Then it is also a root of $(z-1)(a_n z^n + \ldots + a_0) = a_n z^{n+1} + (a_{n-1} - a_n) z^n + \ldots + (a_0 - a_1) z - a_0$. But then we obtain the contradiction

$$\begin{aligned} a_0 &= |a_n z^{n+1} + (a_{n-1} - a_n) z^n + \ldots + (a_0 - a_1) z| \\ &\leq a_n |z|^{n+1} + (a_{n-1} - a_n) |z|^n + \ldots + (a_0 - a_1) |z| \\ &\leq a_n |z| + (a_{n-1} - a_n) |z| + \ldots + (a_0 - a_1) |z| = a_0 |z| < a_0. \end{aligned}$$

We must conclude that all roots of $a_n z^n + \ldots + a_0$ have $|z| \geq 1$.

Remark: This approach is inspired by summation by parts. Define $a_{n+1} = 0$ for convenience. Partial summation yields

$$\begin{aligned} a_n z^n + \ldots + a_0 &= \sum_{k=0}^{n} (a_k - a_{k+1})(z^k + z^{k-1} + \ldots + z + 1) \\ &= \frac{1}{1-z} \sum_{k=0}^{n} (a_k - a_{k+1})(z^{k+1} - 1). \end{aligned}$$

One can now get the idea to multiply the original polynomial by $z - 1$, as is done in the above solution. Alternatively, one can notice that for any $|z| < 1$

$$\mathrm{Re}\left(\sum_{k=0}^{n} (a_k - a_{k+1})(z^{k+1} - 1) \right) = \sum_{k=0}^{n} |a_k - a_{k+1}| \mathrm{Re}(z^{k+1} - 1) < 0,$$

impying that $\sum_{k=0}^{n} (a_k - a_{k+1})(z^{k+1} - 1)$, and hence $a_n z^n + \ldots + a_0$, cannot be 0.

There is yet a different interpretation of the calculations before this remark: They basically show that

$$|(z-1)(a_n z^n + \ldots + a_0) - (-a_0)| < |-a_0|$$

on any circle $|z| = r < 1$. It then follows from Rouché's theorem that $(z-1)(a_n z^n + \ldots + a_0)$, and thus $a_n z^n + \ldots + a_0$, has no roots with $|z| < 1$.

Exercise 4 (*20 pt*): Let f be a meromorphic function on $\mathbb{C}$. Suppose there exist $C, R > 0$ and integer $n \geq 1$ such that $|f(z)| \leq C|z|^n$ for all $z \in \mathbb{C}$ with $|z| \geq R$.

a. (*10 pt*) Prove that the number of poles of f in $\mathbb{C}$ is finite.

We exclude the trivial case $f = 0$. First we prove that f cannot attain infinitely many zeros in the disc $|z| \leq R$, a result we will need later. So suppose f has infinitely many zeros in this disc. Of course this allows us to find a sequence of zeros of f in $|z| \leq D$. Then by compactness of the disc there exists a convergent subsequence with some limit p in $|z| \leq D$. By continuity p must also be a zero of f (it cannot have a pole there, this is because in some neighborhood of p we would then have $|f(z)| \geq C|z-p|^{-m}$ for some positive numbers C and m). However, p is then an accumulation point of a sequence of zero's of f, thus (by Theorem 3.2b, p. 62) f has a power series equal to 0 in a neighborhood of p. By analytic continuation $f = 0$, a contradiction. We must conclude that f has only finitely many zeros.

Now suppose f has infinitely many poles. By the same reasoning as above we can find a convergent sequence of poles of f converging to some limit q. Suppose f has a zero of multiplicity $m \geq 0$ in $z = q$. Then $(z-q)^m/f$ extends to an analytic function in some neighborhood of q, using the fact that f has only finitely many zeros. Exactly analogous to the above this leads to a contradiction. Thus we conclude that f has only finitely many poles in $|z| \leq R$. By the inequality $|f(z)| \leq C|z|^n$ we know that f cannot have poles for $|z| > R$.

b. (*10 pt*) Prove that f is a rational function, i.e. it can be written as a ratio of two polynomials.

Let P be a polynomial containing all the poles (counted with multiplicity) of f. Then Pf extends to an entire function g that satisfies $|g(z)| \leq C|z|^N$ for some number N (the sum of the orders of the poles minus n). Then, for $k > N$ we have using the generalization of Cauchy's Integral Formula that

$$|g^{(k)}(0)| = \left| \frac{k!}{2\pi i} \int_{|z|=r} \frac{g(z)}{z^{k+1}} dz \right| \leq \frac{k! C r^N 2\pi r}{r^{k+1}} = 2\pi k! C r^{N-k}$$

for any $r > R$. Thus we see, by taking the limit $r \to \infty$, that all coefficients of f vanish for $k > N$, i.e. g is a polynomial. This implies that f is a rational function.

Exercise 5 (*25 pt*): Let $a > 0$. By integrating the function

$$f(z) = \frac{1}{z} \frac{1}{\cos(2\pi i a) - \cos(2\pi z)}$$

over a suitable closed path, show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2} = \frac{\pi}{a}\frac{e^{2\pi a}-e^{-2\pi a}}{e^{2\pi a}+e^{-2\pi a}-2}.$$

Hint: Use a square path.

Take the square with vertices $\pm m \pm im$, with m an odd natural number divided by 2. On both its horizontal edges, $t \pm im$, we have

$$|\cos(2\pi ia)-\cos(2\pi z)||z| = |\frac{1}{2}e^{2\pi m}e^{\pm 2\pi it}+\ldots||z| \geq Cme^{2\pi m}$$

for some constant C. Thus the absolute value of these integrals is smaller than or equal to $2m/(Cme^{2\pi m}) = 2/C \cdot e^{-2\pi m}$, which converges to 0 as $m \to \infty$.
For the right vertical edge, $m + imt$, the corresponding integral equals

$$\int_{-1}^{1} \frac{1}{\cosh(2\pi a)+\cosh(2\pi mt)}\frac{idt}{1+it}$$

For every $\epsilon > 0$ we have

$$\lim_{m\to\infty}\left|\int_{\epsilon}^{1} \frac{1}{\cosh(2\pi a)+\cosh(2\pi mt)}\frac{idt}{1+it}\right| \leq \lim_{m\to\infty}\frac{1}{\cosh(2\pi a)+\cosh(2\pi m\epsilon)}\frac{1}{\sqrt{1+\epsilon^2}} = 0$$

This is of course also true for the part from -1 to ϵ. For the middle part we have

$$\left|\int_{-\epsilon}^{\epsilon} \frac{1}{\cosh(2\pi a)+\cosh(2\pi mt)}\frac{idt}{1+it}\right| \leq \frac{2\epsilon}{\cosh(2\pi a)+1}$$

We must conclude that

$$\lim_{m\to\infty}\left|\int_{-1}^{1} \frac{1}{\cosh(2\pi a)+\cosh(2\pi mt)}\frac{idt}{1+it}\right| \leq \frac{2\epsilon}{\cosh(2\pi a)+1} < \epsilon$$

for any $\epsilon > 0$, hence the right vertical integral tends to 0 as $m \to \infty$. The case of the left vertical integral is analogous.
We are now left with the residue at 0 and the residues at $\pm ia+n$. By the Residue Formula we get

$$\begin{aligned}\frac{1}{\cosh(2\pi a)-1} &= \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi\sin(2\pi(ia+n))}\frac{1}{ia+n}+\frac{1}{2\pi\sin(2\pi(-ia+n))}\frac{1}{-ia+n}\right)\\ &= \frac{1}{-2\pi i\sinh(2\pi a)}\sum_{n=-\infty}^{\infty}\frac{-ia+n-(ia+n)}{a^2+n^2}\\ &= \frac{a}{\pi\sinh(2\pi a)}\sum_{n=-\infty}^{\infty}\frac{1}{a^2+n^2}\end{aligned}$$

This implies that

$$\sum_{n=-\infty}^{\infty}\frac{1}{a^2+n^2} = \frac{\pi}{a}\frac{\sinh(2\pi a)}{\cosh(2\pi a)-1} = \frac{\pi}{a}\frac{e^{2\pi a}-e^{-2\pi a}}{e^{2\pi a}+e^{-2\pi a}-2}.$$

Bonus Exercise (*20 pt*): Find all entire functions f such that

$$f(z^2) = (f(z))^2$$

for all $z \in \mathbb{C}$.

Solution 1: Suppose f is not identically zero. We can write $f(z) = z^m g(z)$ for some analytic function g with $g(0) \neq 0$. We notice that g must also satisfy $g(z^2) = g(z)^2$. For all z in the unit disc we have

$$0 \neq |g(0)| = \lim_{n\to\infty} |g(z^{2^n})| = \lim_{n\to\infty} |g(z)|^{2^n}.$$

This limit can only exist and not be equal to 1 if $|g(z)| = 1$ for all z in the unit disc. By the maximum modulus principle this implies that g is constant. In fact $g(0) = g(0)^2$, so we must conclude that $g = 1$. We conclude that the full solution set is given by $f = 0$ and $f(z) = z^m$, m a non-negative integer.

Solution 2: Again we write $f(z) = z^m g(z)$. When $e^{i\phi}$ is a maximum for g on the closed unit disc, so is $e^{i\phi/2}$. Thus the sequence $(e^{i\phi\cdot 2^{-n}})_n$ yields maxima of g. By continuity of $|g|$ a maximum must also be attained in $z = 1$. We know that $g(0) = g(0)^2$ and $g(1) = g(1)^2$. The only possibillity is $g(0) = g(1) = 1$. By the maximum modulus principle g is identically one.

Solution 3: Again we write $f(z) = z^m g(z)$. Since $g(0) \neq 0$ we can find an open ball on which g is non-zero. On this open ball we can define the analytic function

$$h_n(z) = \exp\left(\frac{1}{2^{2^n}} \int_0^z \frac{g'(\zeta)}{g(\zeta)} d\zeta\right)$$

as is done on p.123. We notice (by induction) that

$$g(z) = h_n(z^{2^n}) = f(0) + \frac{g^{(2^n)}(0)}{(2^n)!} z^{2^n} + \dots$$

for all n. This implies that g cannot have a term of smallest positive power in its power series expansion, i.e. it is constant.