ENDTERM COMPLEX FUNCTIONS

JUNE 28, 2016, 8:30-11:30

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.

Exercise 1 (*10 pt*): Determine all entire functions f such that

$$(f(z))^2 + (f'(z))^2 = 1$$

for all $z \in \mathbb{C}$.

Exercise 2 (*10 pt*):

a. (*5 pt*) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a doubly periodic function, i.e., there exist $x_1, x_2 \in \mathbb{C}^*$, no real multiples of each other, such that

$$f(z) = f(z + x_1) = f(z + x_2)$$

for all $z \in \mathbb{C}$. Suppose that f is analytic. Show that f is constant.

b. (*5 pt*) Determine all entire functions f such that the identities

$$f(z+1) = if(z) \qquad \text{and} \qquad f(z+i) = -f(z)$$

hold for all $z \in \mathbb{C}$.

Exercise 3 (*20 pt*):
Prove that the following integrals converge and evaluate them.

a. (*10 pt*) $\displaystyle\int_0^\infty \frac{1}{(x^2 - e^{\pi i/3})^2}\,dx$ **b.** (*10 pt*) $\displaystyle\int_0^\infty \frac{x - \sin x}{x^3}\,dx$

Please turn over!

Exercise 4 (*10 pt*): Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by:

$$f(z) = \begin{cases} e^{-\frac{1}{z^4}} & \text{if } z \neq 0; \\ 0 & \text{if } z = 0. \end{cases}$$

a. (*5 pt*) Show that f satisfies the Cauchy-Riemann equations on the whole of $\mathbb{C}$.

b. (*5 pt*) Is f analytic? Motivate your answer.

Exercise 5 (*10 pt*):
Let f be an entire function that sends the real axis to the real axis and the imaginary axis to the imaginary axis. Show that f is an odd function.

Exercise 6 (*20 pt*):
Let $U \subseteq \mathbb{C}$ be a connected open set. Let $\{f_n\}$ be a sequence of complex functions on U which converges uniformly on every compact subset of U to the limit function f. (I.e., for every compact subset K of U, $\{f_n|K\}$ converges uniformly on K to $f|K$.)

a. (*5 pt*) Give an example where the f_n are injective and holomorphic, but f is constant.

b. (*5 pt*) Give an example where the f_n are injective and (real) differentiable, but f is neither constant nor injective.

Hint: When is $z \mapsto z + a\bar{z}$ injective? Holomorphic?

c. (*10 pt*) Prove: if the f_n are injective and holomorphic, then f is either constant or injective.

Hint 1: Reduce the problem to the following special case: If $f(z_0) = f(z_1) = 0$, with $z_0 \neq z_1$, and $f_n(z_0) = 0$ for all n, then $f \equiv 0$.

Hint 2: Now look at the orders of f and the f_n at z_1.