Instructor: Carel Faber
Assistants: Stefan Franssen, Bjarne Kosmeijer, Roy Wang

SOLUTIONS ENDTERM COMPLEX FUNCTIONS

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Exercise 1 (*10 pt*): Determine all entire functions f such that

$$(f(z))^2 + (f'(z))^2 = 1$$

for all $z \in \mathbb{C}$.

Solution. Taking the derivative, we find

$$0 = 2f(z)f'(z) + 2f'(z)f''(z) = 2f'(z)(f(z) + f''(z)).$$

If $g(z)h(z) = 0$ on an infinite compact set, then the zeroes of g or h form an infinite set with a point of accumulation. If g and h are moreover analytic, then $g \equiv 0$ or $h \equiv 0$. We conclude that $f' \equiv 0$ or $f + f'' \equiv 0$. In the first case, f is constant; hence $f \equiv 1$ or $f \equiv -1$. In the second case, we know (e.g., by Exercise 6 in §II.6) that there is a unique solution with given initial conditions. Of course $\cos(z)$ and $\sin(z)$ are solutions, so $f(z) = a\cos(z) + b\sin(z)$ is the unique solution with $f(0) = a$ and $f'(0) = b$. Finally, $a\cos(z) + b\sin(z)$ satisfies the original equation if and only if a and b are complex numbers with $a^2 + b^2 = 1$. Answer: $f \equiv 1$ or $f \equiv -1$ or $f(z) = a\cos(z) + b\sin(z)$, with $a^2 + b^2 = 1$.

Exercise 2 (*10 pt*):

a. (*5 pt*) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a doubly periodic function, i.e., there exist $x_1, x_2 \in \mathbb{C}^*$, no real multiples of each other, such that

$$f(z) = f(z + x_1) = f(z + x_2)$$

for all $z \in \mathbb{C}$. Suppose that f is analytic. Show that f is constant.

Solution. Let K be the parallelogram with vertices 0, x_1, x_2 and $x_1 + x_2$, i.e.,

$$K = \{z = t_1x_1 + t_2x_2 \,|\, 0 \le t_1,\, t_2 \le 1\}.$$

From the double periodicity, it follows that for all $z \in \mathbb{C}$ there exists $w \in K$ with $f(w) = f(z)$ (note that K and its translates over integral linear combinations of x_1 and x_2 tile the plane). Since f is continuous and K is compact, f is bounded on K. Hence f is bounded. By Liouville's theorem, we conclude that f is constant.

b. (*5 pt*) Determine all entire functions f such that the identities

$$f(z+1) = if(z) \qquad \text{and} \qquad f(z+i) = -f(z)$$

hold for all $z \in \mathbb{C}$.

Solution. Note that $f(z+4) = if(z+3) = \cdots = i^4 f(z) = f(z)$ and $f(z+2i) = -f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Hence f is doubly periodic and by part (a), f is constant: $f \equiv c$ for some $c \in \mathbb{C}$. Since $c = ic$, we find that f is identically equal to zero.

Exercise 3 (*20 pt*):

Prove that the following integrals converge and evaluate them.

a. (*10 pt*) $\displaystyle\int_0^\infty \frac{1}{(x^2 - e^{\pi i/3})^2}\, dx$ **b.** (*10 pt*) $\displaystyle\int_0^\infty \frac{x - \sin x}{x^3}\, dx$

Solution of part (a). Convergence follows from an estimate like $|x^2 - e^{\pi i/3}| \geq |x^2 - 1| \geq x^2/2$ for $x > 2$. Note that the integrand is even. We integrate $f(z) = 1/(z^2 - e^{\pi i/3})^2$ over a contour consisting of the segment from $-R$ to R and the counterclockwise semicircle $S(R)$ around 0 from R to $-R$, for $R > 1$ large enough. Note that $|z^2 - e^{\pi i/3}| \geq R^2 - 1$ when $|z| = R$, so $|\int_{S(R)} f(z)dz| \leq \pi R/(R^2-1)^2$, so $\int_{S(R)} f(z)dz \to 0$ as $R \to \infty$. Next, the poles of f are at the points z with $z^2 = e^{\pi i/3}$, i.e., at $z = \pm e^{\pi i/6}$; the only pole in the upper half plane is at $\alpha = e^{\pi i/6}$, inside the contour. Now

$$\mathrm{Res}_\alpha(f) = \mathrm{Res}_\alpha \frac{1}{(z-\alpha)^2(z+\alpha)^2} = \left.\frac{-2}{(z+\alpha)^3}\right|_{z=\alpha} = \frac{-2}{8\alpha^3} = \frac{-1}{4i}.$$

It follows that $\int_{-\infty}^\infty \frac{1}{(x^2-e^{\pi i/3})^2}\, dx = \frac{-2\pi i}{4i} = -\frac{\pi}{2}$, so $\int_0^\infty \frac{1}{(x^2-e^{\pi i/3})^2}\, dx = -\frac{\pi}{4}$.

Solution of part (b). The integrand can be continuously extended to the origin. Convergence at infinity follows from an estimate like $|(x - \sin x)/x^3| \leq 2/x^2$ for $x > 1$. Let $f(z) = (iz - e^{iz})/z^3$. For $0 < \epsilon < R$, we have $\int_{-R}^{-\epsilon} f(x)\, dx = \int_\epsilon^R (ix + e^{-ix})/x^3\, dx$, so

$$2i\int_\epsilon^R (x - \sin x)/x^3\, dx = \int_\epsilon^R (2ix - e^{ix} + e^{-ix})/x^3\, dx =$$

$$\int_\epsilon^R (ix - e^{ix})/x^3\, dx + \int_\epsilon^R (ix + e^{-ix})/x^3\, dx = \int_\epsilon^R f(x)\, dx + \int_{-R}^{-\epsilon} f(x)\, dx.$$

We integrate f over a contour consisting of segments from $-R$ to $-\epsilon$ and from ϵ to R and semicircles around 0 in the upper half plane from $-\epsilon$ to ϵ and from R to $-R$. The integral over the contour is zero. The integral over $S(R)$ goes to zero as $R \to \infty$, since $|e^{iz}| \leq 1$ for z in the upper half plane. The integral over the counterclockwise semicircle $S(\epsilon)$ can be evaluated by means of integration by parts:

$$\int_{S(\epsilon)} \frac{iz - e^{iz}}{z^3}\, dz = -\frac{1}{2}\left.\frac{iz - e^{iz}}{z^2}\right|_\epsilon^{-\epsilon} + \frac{1}{2}\int_{S(\epsilon)} \frac{i - ie^{iz}}{z^2}\, dz.$$

As $\epsilon \to 0$, the limit equals

$$\frac{1}{2}\pi i(-i \cdot i) - \frac{1}{2}\lim_{\epsilon\to 0}\left(\frac{-i\epsilon - e^{-i\epsilon}}{\epsilon^2} - \frac{i\epsilon - e^{i\epsilon}}{\epsilon^2}\right)$$
$$= \frac{1}{2}\pi i - \frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\big(- i\epsilon - 1 + i\epsilon - \frac{1}{2}(i\epsilon)^2 - i\epsilon + 1 + i\epsilon + \frac{1}{2}(i\epsilon)^2\big) = \frac{1}{2}\pi i.$$

Hence the limit as $R \to \infty$ and $\epsilon \to 0$ of the integral of $f(x)$ over the two segments equals $\frac{1}{2}\pi i$ and $\int_0^\infty \frac{x - \sin x}{x^3}\, dx = \frac{1}{2}\pi i/(2i) = \pi/4$.

Exercise 4 (*10 pt*): Let $f : \mathbb{C} \to \mathbb{C}$ be defined by:

$$f(z) = \begin{cases} e^{-\frac{1}{z^4}} & \text{if } z \neq 0; \\ 0 & \text{if } z = 0. \end{cases}$$

a. (*5 pt*) Show that f satisfies the Cauchy-Riemann equations on the whole of $\mathbb{C}$.

Solution. Since f is holomorphic on $\mathbb{C}\backslash\{0\}$, it satisfies the C-R equations there. Next, note that if z is real or imaginary, then z^4 is real, so f is real along the real and imaginary axes. So $v_x(0,0) = v_y(0,0) = 0$ and $u_x(0,0) = \lim_{x\to 0}\frac{u(x,0)-0}{x} = \lim_{x\to 0}(e^{-1/x^4})/x = 0 = \lim_{y\to 0}(e^{-1/y^4})/y = \lim_{y\to 0}\frac{u(0,y)-0}{y} = u_y(0,0)$, hence the C-R equations hold at the origin as well. (We used that $(iy)^4 = y^4$.)

b. (*5 pt*) Is f analytic? Motivate your answer.

Solution. Taking $z = te^{i\pi/4}$, we have $z^4 = -t^4$, so $\lim_{t\to 0} f(z) = +\infty$, so f is not even continuous at 0. Alternatively, the restriction of f to $\mathbb{C}\backslash\{0\}$ admits a Laurent expansion at 0 with infinitely many negative terms, so 0 is an essential singularity of the restriction, not a removable one.

Exercise 5 (*10 pt*):

Let f be an entire function that sends the real axis to the real axis and the imaginary axis to the imaginary axis. Show that f is an odd function.

Solution. First, $f(0) \in \mathbb{R} \cap i\mathbb{R}$, so $f(0) = 0$. Put $g(z) = f(z) + f(-z)$; we need to show that $g \equiv 0$. The power series expansion for g at 0 is of the form $\sum_{k=1}^{\infty} a_{2k}z^{2k}$ and converges everywhere. Assume that $g \not\equiv 0$; let $m > 0$ be minimal such that $a_{2m} \neq 0$. Then $g(z) = a_{2m}z^{2m}(1 + h(z))$, where $h(z)$ is a convergent power series without constant term, thus $|h(z)|$ is small for $|z|$ small enough. In particular, $|\arg(1+h(z))|$ is small for $|z|$ small enough. Substituting $z = r$ with r a small nonzero real number, we find that a_{2m} is approximately real; but substituting $z = ir$, we find that a_{2m} is approximately imaginary. This is a contradiction, so $g \equiv 0$, so f is odd. (Approximately real means $a_{2m} = Re^{i\phi}$

with $-t < \phi < t$ or $\pi - t < \phi < \pi + t$ for some small $t > 0$; approximately imaginary means $a_{2m} = Re^{i\phi}$ with $\pi/2 - t < \phi < \pi/2 + t$ or $3\pi/2 - t < \phi < 3\pi/2 + t$.)

Exercise 6 (*20 pt*):

Let $U \subseteq \mathbb{C}$ be a connected open set. Let $\{f_n\}$ be a sequence of complex functions on U which converges uniformly on every compact subset of U to the limit function f. (I.e., for every compact subset K of U, $\{f_n|K\}$ converges uniformly on K to $f|K$.)

a. (*5 pt*) Give an example where the f_n are injective and holomorphic, but f is constant.

Solution. Take $f_n(z) = z/n$, for example. Then $f \equiv 0$. Given K compact, there exists $R > 0$ such that $|z| \leq R$ for all $z \in K$. So for $n > N := R/\epsilon$ we have that $||f_n - f||_K < \epsilon$. Moreover, the f_n are injective and holomorphic, but f is constant.

b. (*5 pt*) Give an example where the f_n are injective and (real) differentiable, but f is neither constant nor injective.

Hint: When is $z \mapsto z + a\bar{z}$ injective? Holomorphic?

Solution. We note that $z \mapsto z + a\bar{z}$ is holomorphic exactly when $a = 0$. Assume $z_1 \neq z_2$. They have the same image when $(z_1 - z_2) + a\overline{(z_1 - z_2)} = 0$. This implies $|a| = 1$ and, conversely, when $|a| = 1$, there exist $z_1 \neq z_2$ with the same image. So $z \mapsto z + a\bar{z}$ is injective exactly when $|a| \neq 1$. Take $f_n(z) = z + (1 + 1/n)\bar{z}$, converging uniformly on compact subsets to $f(z) = z + \bar{z}$. Then the f_n are injective and real differentiable, but f is neither constant nor injective.

c. (*10 pt*) Prove: if the f_n are injective and holomorphic, then f is either constant or injective.

Hint 1: Reduce the problem to the following special case: If $f(z_0) = f(z_1) = 0$, with $z_0 \neq z_1$, and $f_n(z_0) = 0$ for all n, then $f \equiv 0$.

Hint 2: Now look at the orders of f and the f_n at z_1.

Solution. Suppose f is not injective. Then there exist $z_0 \neq z_1$ with $f(z_0) = f(z_1)$. Assume $f_n \to f$, uniformly on compact subsets. Then $f_n(z_0) \to f(z_0)$. Subtracting $f_n(z_0)$ from f_n and $f(z_0)$ from f, we may assume $f_n(z_0) = 0$; and the new f_n converge to the new f, uniformly on compact subsets. We know $f(z_0) = f(z_1) = 0$ and should prove $f \equiv 0$. This accomplishes the suggested reduction.

Suppose that $f \not\equiv 0$. Then f is not locally constant near z_1, since U is (open and) connected. So the order of f at z_1 is positive, say $m > 0$. Then we know that there exists a suitable local coordinate w at z_1 such that $f(w) = w^m$ in a neighborhood $V \subset U$ of z_1.

Choose $r > 0$ so that the closed disc $D = \{|w| \leq r\}$ is contained in V and doesn't contain z_0. Choose $\epsilon > 0$ with $\epsilon < r^m$. Choose n such that $||f_n - f||_D < \epsilon$. Rouché's theorem gives us that f_n and f have the same number of zeros inside $\{|w| = r\}$, i.e., at least m when counted with multiplicity. So f_n has a zero other than z_0, contradicting injectivity. This proves that $f \equiv 0$.

Alternatively, staying closer to the second hint, z_1 is an isolated zero of f, hence $|f(z)|$ has a positive lower bound on a small enough circle γ around z_1, so that $1/f_n \to 1/f$, $f_n' \to f'$, and $f_n'/f_n \to f'/f$, all convergences uniform on γ. Then

$$\operatorname{ord}_{z_1} f_n = \frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz;$$

on the one hand, this equals zero, but on the other hand, it converges to $\operatorname{ord}_{z_1} f$, which is positive, as $n \to \infty$; a contradiction again.