RETAKE COMPLEX FUNCTIONS

JULY 21, 2016, 13:30-16:30

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.
- No auxiliary material allowed.

Exercise 1 (*15 pt*): Determine all entire functions f such that

$$(f(z))^2 - (f'(z))^2 = 1$$

for all $z \in \mathbb{C}$.

Exercise 2 (*30 pt*):
Prove that the following integrals converge and evaluate them.

$$\textbf{a.}\ (\textit{15 pt}) \int_0^\infty \frac{1}{(x^2+1)^3}\,dx \qquad \textbf{b.}\ (\textit{15 pt}) \int_0^\infty \frac{\log x}{x^4+1}\,dx$$

(*Hint for (b): Use a contour consisting of two semicircles and two segments and use an appropriate definition of the complex logarithm.*)

Exercise 3 (*15 pt*): Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that $f(1) = 2f(0)$. Given $\epsilon > 0$, prove that there exists $z \in \mathbb{C}$ with $|f(z)| < \epsilon$.

Please turn over!

Exercise 4 (*15 pt*): Consider the polynomial function $f(z) = z^3 + Az^2 + B$, where A and B are complex numbers. Assume that the following inequalities hold:

$$|A| + 1 < |B| < 4|A| - 8.$$

a. (*10 pt*) Determine the number of zeroes (counted with multiplicities) of $f(z)$ with $|z| \leq 1$ and also the number of zeroes (with multiplicities) of $f(z)$ with $|z| \leq 2$.

b. (*5 pt*) By finding the zeroes of $z^3 - 3z^2 + 4$, show that these numbers of zeroes (with multiplicities) may be different when

$$|A| + 1 = |B| = 4|A| - 8.$$

Exercise 5 (*15 pt*): Let f be an entire function that sends both the real axis and the imaginary axis to the real axis.

a. (*5 pt*) Give an example of such a function for which in addition the following two properties hold:

(i) f is surjective;

(ii) $f(\mathbb{R}) \cap f(i\mathbb{R}) = \{f(0)\}$.

b. (*10 pt*) Prove that no function satisfying the original hypotheses is injective. (I.e., you should prove: if f is entire and $f(\mathbb{R}) \subseteq \mathbb{R}$ and $f(i\mathbb{R}) \subseteq \mathbb{R}$, then f is not injective.)

Bonus Exercise (*15 pt*): Assume that f is analytic in the punctured disc $\{z \in \mathbb{C} \,|\, 0 < |z| < R\}$ of radius $R > 0$ and that the isolated singularity of f at $z = 0$ is not removable. Prove that $g(z) = \exp(f(z))$ has an essential singularity at $z = 0$.
Hint: There are two cases: f has a pole at $z = 0$ or an essential singularity. When f has a pole, use a suitable local coordinate.