Instructor: Carel Faber
Assistants: Stefan Franssen, Bjarne Kosmeijer, Roy Wang

SOLUTIONS RETAKE COMPLEX FUNCTIONS

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Exercise 1 (*15 pt*): Determine all entire functions f such that

$$(f(z))^2 - (f'(z))^2 = 1$$

for all $z \in \mathbb{C}$.

Solution. Taking the derivative, we find

$$0 = 2f(z)f'(z) - 2f'(z)f''(z) = 2f'(z)(f(z) - f''(z)).$$

If $g(z)h(z) = 0$ on an infinite compact set, then the zeroes of g or h form an infinite set with a point of accumulation. If g and h are moreover analytic, then $g \equiv 0$ or $h \equiv 0$. We conclude that $f' \equiv 0$ or $f - f'' \equiv 0$. In the first case, f is constant; hence $f \equiv 1$ or $f \equiv -1$. In the second case, we know (e.g., by Exercise 6 in §II.6) that there is a unique solution with given initial conditions. Of course $\exp(z)$ and $\exp(-z)$ are solutions, so $f(z) = a\exp(z) + b\exp(-z)$ is the unique solution with $f(0) = a + b$ and $f'(0) = a - b$. Finally, $a\exp(z) + b\exp(-z)$ satisfies the original equation if and only if a and b are complex numbers with $4ab = 1$. Answer: $f \equiv 1$ or $f \equiv -1$ or $f(z) = a\exp(z) + b\exp(-z)$, with $4ab = 1$.

Exercise 2 (*30 pt*):
Prove that the following integrals converge and evaluate them.

$$\textbf{a.}\ (\textit{15 pt})\ \int_0^\infty \frac{1}{(x^2+1)^3}\,dx \qquad \textbf{b.}\ (\textit{15 pt})\ \int_0^\infty \frac{\log x}{x^4+1}\,dx$$

(*Hint for (b): Use a contour consisting of two semicircles and two segments and use an appropriate definition of the complex logarithm.*)

Solution of part (a). The integrand is continuous on $\mathbb{R}$ and convergence at infinity follows from $1/(x^2+1)^3 \le 1/x^2$. Note that the integrand is even. We integrate $f(z) = 1/(z^2+1)^3$ over a contour consisting of the segment from $-R$ to R and the counterclockwise semicircle $S(R)$ around 0 from R to $-R$, for $R > 1$ large enough. Note that $|z^2+1| \ge R^2-1$ when $|z| = R$, so $|\int_{S(R)} f(z)dz| \le \pi R/(R^2-1)^3$, so $\int_{S(R)} f(z)dz \to 0$ as $R \to \infty$. Next, the poles of f are at the points z with $z^2 = -1$, i.e., at $z = \pm i$; the only pole in the upper half plane is at i, inside the contour. Now, from the Taylor expansion at $z = i$,

$$\mathrm{Res}_i(f) = \mathrm{Res}_i \frac{1}{(z-i)^3(z+i)^3} = \frac{1}{2}\left(\frac{1}{(z+i)^3}\right)''\bigg|_{z=i} = \frac{1}{2}\left(\frac{-3}{(z+i)^4}\right)'\bigg|_{z=i} = \frac{1}{2}\frac{12}{(z+i)^5}\bigg|_{z=i} = \frac{6}{32i} = \frac{3}{16i}.$$

It follows that $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\, dx = \frac{6\pi i}{16 i} = \frac{3\pi}{8}$, so $\int_0^{\infty} \frac{1}{(x^2+1)^3}\, dx = \frac{3\pi}{16}$.

Solution of part (b). Near zero, the integrand is bounded in absolute value by $-\log x$ and $\int_0^1 -\log x\, dx = (x - x\log x)\Big|_0^1 = 1$, since $\lim_{x\to 0} x\log x = 0$, so the integral converges near zero. Convergence near infinity follows from an estimate like $|(\log x)/(x^4+1)| < 1/x^3$ for $x > 1$. Following the hint, we take a contour consisting of the counterclockwise semicircle $S(R)$ around 0 from R to $-R$, for $R > 1$ large enough, the clockwise semicircle $-S(\delta)$ around 0 from $-\delta$ to δ, for $0 < \delta < 1/2$ small enough, and the segments from $-R$ to $-\delta$ and from δ to R. Let $f(z) = (\log z)/(z^4+1)$, where we take the (branch of the) complex logarithm on $\mathbb{C} \setminus i\mathbb{R}_{\leq 0}$ that continues $\log x$ for $x > 0$, i.e., for $z = r\exp(i\phi)$ with $r > 0$ and $-\pi/2 < \phi < 3\pi/2$ we have $\log z = \log r + i\phi$. Then $\int_{-R}^{-\delta} f(y)\, dy = [y = -x,\ dy = -dx,\ \log y = \log x + i\pi] = \int_\delta^R (\log x + i\pi)/(x^4+1)\, dx$.
Now $\int_{S(R)} f(z)\, dz = \int_0^\pi \frac{\log R + i\phi}{R^4\exp(4i\phi)+1} iR\exp(i\phi)\, d\phi$, so $|\int_{S(R)} f(z)\, dz| \leq \pi R \frac{\log R + \pi}{R^4 - 1}$, which goes to 0 when $R \to \infty$. Similarly, $\int_{S(\delta)} f(z)\, dz = \int_0^\pi \frac{\log \delta + i\phi}{\delta^4 \exp(4i\phi)+1} i\delta\exp(i\phi)\, d\phi$, so $|\int_{S(\delta)} f(z)\, dz| \leq \pi\delta\frac{16}{15}(|\log\delta| + \pi)$, which again goes to 0 as $\delta \to 0$.
The integrand has simple poles at the four zeroes of z^4+1. Put $\alpha = \exp(i\pi/4)$, then the poles in the upper half plane are at α and at α^3. Now $\mathrm{Res}_\alpha(f) = (\log\alpha)\lim_{z\to\alpha}\frac{z-\alpha}{z^4+1} = i\pi/4\frac{1}{4\alpha^3} = \frac{i\pi\alpha^5}{16} = \frac{-i\pi\alpha}{16}$ and $\mathrm{Res}_{\alpha^3}(f) = (\log(\alpha^3))\lim_{z\to\alpha^3}\frac{z-\alpha^3}{z^4+1} = 3i\pi/4\frac{1}{4\alpha^9} = \frac{3i\pi\alpha^7}{16} = \frac{-3i\pi\alpha^3}{16}$, so the sum of the residues equals $\frac{i\pi}{16}(-\alpha - 3\alpha^3) = \frac{i\pi}{16}(-(1+i)/\sqrt{2} - 3(-1+i)/\sqrt{2}) = \frac{i\pi}{16\sqrt{2}}(-1 - i + 3 - 3i) = \frac{i\pi}{16\sqrt{2}}(2-4i)$. The integral to be computed equals $1/2(2\pi i)\frac{i\pi}{16\sqrt{2}}2 = -\frac{\pi^2}{8\sqrt{2}} = -\frac{\pi^2\sqrt{2}}{16}$.
Along the way, we have also computed $\int_0^\infty \frac{i\pi}{x^4+1}\, dx = (2\pi i)\frac{i\pi}{16\sqrt{2}}(-4i) = i\frac{\pi^2}{2\sqrt{2}} = i\frac{\pi^2\sqrt{2}}{4}$, which also can be computed more directly, of course.

Exercise 3 (*15 pt*): Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that $f(1) = 2f(0)$. Given $\epsilon > 0$, prove that there exists $z \in \mathbb{C}$ with $|f(z)| < \epsilon$.

Solution. Let $\epsilon > 0$ be given and assume there is no $z \in \mathbb{C}$ with $|f(z)| < \epsilon$. Then f certainly doesn't have zeroes. It follows that f is not constant, for the only constant function f with $f(1) = 2f(0)$ is the zero function. Also, $g(z) = 1/f(z)$ is entire; and $|g(z)| \leq 1/\epsilon$ for all $z \in \mathbb{C}$, so g is bounded and entire, hence constant by Liouville's theorem; but then f is constant and we have reached a contradiction. Q.E.D.

Exercise 4 (*15 pt*): Consider the polynomial function $f(z) = z^3 + Az^2 + B$, where A and B are complex numbers. Assume that the following inequalities hold:

$$|A| + 1 < |B| < 4|A| - 8.$$

a. (*10 pt*) Determine the number of zeroes (counted with multiplicities) of $f(z)$ with $|z| \leq 1$ and also the number of zeroes (with multiplicities) of $f(z)$ with $|z| \leq 2$.

Solution. We apply Rouché's theorem. Note that $|A| > 3$, so $|B| > 4$. With $g(z) = B$, $f(z) - g(z) = z^3 + Az^2$, so when $|z| = 1$, $|f(z) - g(z)| \le |A| + 1 < |B| = |g(z)|$, so $f(z)$ and $g(z)$ have no zeroes when $|z| = 1$ and $f(z)$ and $g(z)$ have the same number of zeroes in $\{|z| < 1\}$, i.e., none, since $B \neq 0$. So $f(z)$ has **no** zeroes with $|z| \le 1$.

Next, with $g(z) = Az^2 + B$, $f(z) - g(z) = z^3$, so when $|z| = 2$, $|f(z) - g(z)| = 8 < 4|A| - |B| \le |g(z)|$, so $f(z)$ and $g(z)$ have no zeroes when $|z| = 2$ and $f(z)$ and $g(z)$ have the same number of zeroes (with multiplicities) in $\{|z| < 2\}$, i.e., two, since $|B| < 4|A|$ and $|A| > 0$, so $|B/A| < 4$, so $|\pm\sqrt{-B/A}| < 2$. So $f(z)$ has **two** zeroes with $|z| \le 2$, counted with multiplicity.

b. (*5 pt*) By finding the zeroes of $z^3 - 3z^2 + 4$, show that these numbers of zeroes (with multiplicities) may be different when

$$|A| + 1 = |B| = 4|A| - 8.$$

Solution. Trying for rational solutions, we recall that these should be integers dividing 4; we see that -1 and 2 are zeroes. The sum of all three zeroes equals 3, so 2 is a double zero. So the number of zeroes with $|z| \le 1$ equals **one** and the number of zeroes (with multiplicities) with $|z| \le 2$ equals **three**. (Of course $A = -3$ and $B = 4$ satisfy the two equalities.)

Another way to find the solutions is to try for a double zero; it should satisfy $3z^2 = 6z$, which leads to $z = 2$.

Yet another way is to realize that the 'extra' zeroes should have absolute values 1, respectively 2. Since there is at least one real zero, one tries ± 1 and ± 2 and solves the remaining equation (of degree ≤ 2) if necessary.

Exercise 5 (*15 pt*): Let f be an entire function that sends both the real axis and the imaginary axis to the real axis.

a. (*5 pt*) Give an example of such a function for which in addition the following two properties hold:

(i) f is surjective;

(ii) $f(\mathbb{R}) \cap f(i\mathbb{R}) = \{f(0)\}$.

Solution. An example is $f(z) = z^2$. Then $f(\mathbb{R}) = \mathbb{R}_{\ge 0}$ and $f(i\mathbb{R}) = \mathbb{R}_{\le 0}$. Also, $f(\sqrt{r}\exp(i\phi/2)) = r\exp(i\phi)$, so f is surjective. Another example is $f(z) = -10z^6$, etc.

b. (*10 pt*) Prove that no function satisfying the original hypotheses is injective. (I.e., you should prove: if f is entire and $f(\mathbb{R}) \subseteq \mathbb{R}$ and $f(i\mathbb{R}) \subseteq \mathbb{R}$, then f is not injective.)

Solution. Since $f(0) \in \mathbb{R}$, we can replace f by $f - f(0)$, i.e., we may and will assume $f(0) = 0$. (Segments of) the real axis and the imaginary axis form examples of two curves through the origin with angle $\pi/2$. By Theorem I.7.1, we know that if $f'(0) \neq 0$, then the angle between $f(\mathbb{R})$ and $f(i\mathbb{R})$ at $f(0) = 0$ also equals $\pi/2$. But $f(\mathbb{R})$ and $f(i\mathbb{R})$ are both contained in $\mathbb{R}$, so the angle is not $\pi/2$, so $f'(0) = 0$. From Theorem II.6.4, it now follows that f is not injective on any open neighbourhood of the origin.

Bonus Exercise (*15 pt*): Assume that f is analytic in the punctured disc $\{z \in \mathbb{C} \,|\, 0 < |z| < R\}$ of radius $R > 0$ and that the isolated singularity of f at $z = 0$ is not removable. Prove that $g(z) = \exp(f(z))$ has an essential singularity at $z = 0$.
Hint: There are two cases: f has a pole at $z = 0$ or an essential singularity. When f has a pole, use a suitable local coordinate.

Solution. If f has a pole at $z = 0$ of order $m \geq 1$, then $h = 1/f$ has a zero at $z = 0$ of order m. We then know that there exists a local analytic coordinate w in a neighbourhood of 0 such that $h(w) = w^m$. Then $f(w) = 1/w^m$ and $g(w) = \sum_{n=0}^{\infty} w^{-mn}/n!$. This is the Laurent expansion (in w) of g near 0; it has infinitely many negative terms, so g has an essential singularity at $w = 0$, i.e., at $z = 0$.

If f has an essential singularity at $z = 0$, then by the Casorati-Weierstrass theorem, $f(U)$ is dense in $\mathbb{C}$ for every open neighbourhood U of 0. Then $g(U)$ is dense in $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ (write it out), hence in $\mathbb{C}$, which is more than enough to conclude that g has an essential singularity at $z = 0$.