RETAKE COMPLEX FUNCTIONS

AUGUST 21 2013, 9:00-12:00

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.

Exercise 1 ***(15 pt)*** Let $f(z) = a\bar{z} + b$ where $a, b \in \mathbb{C}$ with $|a| = 1$. Assume that $z \mapsto f(z)$ defines a reflection w.r.t. some line in the complex plane and find the equation of this line.

Exercise 2 ***(15 pt)*** Find the convergence radius of the series

$$\sum_{n=1}^{\infty} z^{n!},$$

where $n! = 1 \cdot 2 \cdots (n-1) \cdot n$.

Exercise 3 ***(20 pt)*** Let U be a simply connected open set. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of injective analytic functions on U that converges uniformly to f . Prove that f is either constant or injective.
Hint: Use Rouché Theorem.

Exercise 4 ***(50 pt)*** Let $0 < a < b$ and let

$$I := \frac{1}{\pi} \int_a^b \frac{\sqrt{(x-a)(b-x)}}{x}\, dx\ . \tag{1}$$

a. ***(10 pt)*** Prove that there exists an analytic function $g : \mathbb{C} \setminus [a, b] \to \mathbb{C}$ such that $[g(z)]^2 = (z-a)(b-z)$ for $z \in \mathbb{C} \setminus [a, b]$ while

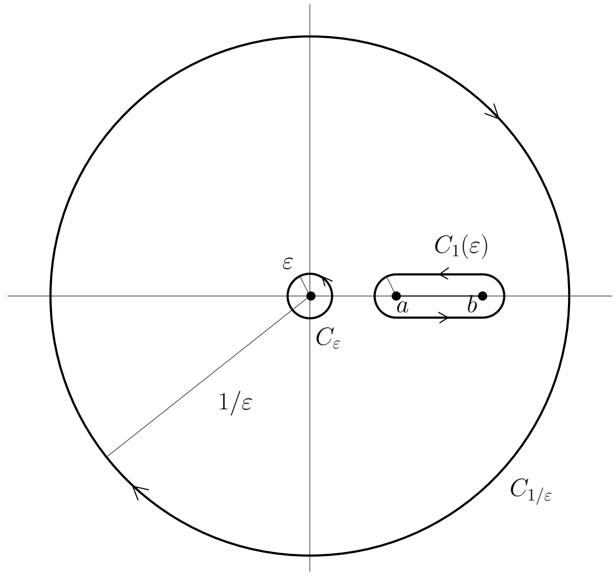
$$\begin{aligned} \lim_{\varepsilon \uparrow 0} g(x + i\varepsilon) &= \sqrt{(x-a)(b-x)}, \\ \lim_{\varepsilon \downarrow 0} g(x + i\varepsilon) &= -\sqrt{(x-a)(b-x)} \end{aligned}$$

for $x \in [a, b]$.

b. ***(10 pt)*** Consider the integral

$$\int_C \frac{g(z)}{z} dz$$

over the closed chain $C = C_\varepsilon + C_{1/\varepsilon} + C_1(\varepsilon)$ shown below



with ε small enough. Argue that this integral vanishes.

c. ***(10 pt)*** Evaluate the integrals over the circles

$$\int_{C_\varepsilon} \frac{g(z)}{z} dz \quad \text{and} \quad \int_{C_{1/\varepsilon}} \frac{g(z)}{z} dz$$

using residues. *Hint*: Substitute $w = 1/z$ in the second integral.

d. ***(10 pt)*** Prove that

$$\lim_{\varepsilon \downarrow 0} \int_{C_1(\varepsilon)} \frac{g(z)}{z} dz = 2\pi I,$$

where $C_1(\varepsilon)$ is the path near the segment $[a, b]$ and I is the integral (1).

e. ***(10 pt)*** Combine the obtained results to explicitly evaluate I.