ENDTERM COMPLEX FUNCTIONS

JUNE 27, 2017, 9:00-12:00

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.

Notation: For $a \in \mathbb{C}$ and $r > 0$, we write $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$, and $\overline{D}(a,r)$ and $C(a,r)$ are the closure and boundary respectively of $D(a,r)$.

Exercise 1 (*10 pt*):
Evaluate the following integral (which clearly is convergent).

$$\int_0^\infty \frac{1}{(x^2+4)(x^2+9)}\, dx.$$

Exercise 2 (*15 pt*):

Fix $R > 0$ and $a \in \mathbb{C}$; we write $D := D(a,R)$ and $\overline{D} := \overline{D}(a,R)$ and $C := C(a,R)$. Let $f, g : \overline{D} \to \mathbb{C}$ be continuous functions, analytic on D, such that $|f(z)| = |g(z)|$ for all $z \in C$, and such that f and g have no zeros in $\overline{D}$. Show that $f = \alpha g$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

Exercise 3 (*15 pt*):

Let $a, b \in \mathbb{C}$. Consider the polynomial $p(z) = z^7 + az^4 + bz^2 - 2$.

(a) Show that if $|z| \leq 1/\sqrt{2}$, then

$$|p(z)| \geq \frac{32 - \sqrt{2} - 4|a| - 8|b|}{16}.$$

(b) Suppose that

$$|b| + 3 < |a| \leq \tfrac{15}{2} - 2|b|.$$

Show that, counting zeros with their multiplicities, p has

(i) no zeros in the disk $|z| \leq 1/\sqrt{2}$,
(ii) four zeros in the annulus $1/\sqrt{2} < |z| < 1$,
(iii) three zeros in the annulus $1 < |z| < 2$,
(iv) and no zeros in the annulus $2 \leq |z|$.

Exercise 4 (*15 pt*): Let

$$f(z) = \frac{z^2(z-1)e^z}{\sin^2 \pi z}$$

and let $U \subset \mathbb{C}$ be the domain of f. Let $V \subset \mathbb{C}$ be the maximal open set on which a holomorphic function g can be defined that agrees with f on U. For each $v \in V$, determine the radius of convergence of the power series for g at v.

Exercise 5 (*15 pt*):
Let f be a non-constant entire function. Prove that the closure of $f(\mathbb{C})$ equals $\mathbb{C}$.

Exercise 6 (*20 pt*):
Prove that the following integral converges and evaluate it.

$$\int_0^\infty \frac{\log x}{x^3+1}\, dx.$$

(*Hint: Use a contour consisting of two circular arcs and two segments, with 'vertices' ϵ, R, Rc, and ϵc, where $c^3 = 1$, $c \neq 1$. Use the natural substitution to relate the integrals over the two segments. Use an appropriate definition of the complex logarithm.*)