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SOLUTIONS ENDTERM COMPLEX FUNCTIONS

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Notation: For $a \in \mathbb{C}$ and $r > 0$, we write $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$, and $\overline{D}(a,r)$ and $C(a,r)$ are the closure and boundary respectively of $D(a,r)$.

Exercise 1 (*10 pt*):
Evaluate the following integral (which clearly is convergent).

$$\int_0^\infty \frac{1}{(x^2+4)(x^2+9)}\,dx.$$

Solution. Note that the integrand is even. We integrate $f(z) = 1/((z^2+4)(z^2+9))$ over a contour consisting of the segment from $-R$ to R and the counterclockwise semicircle $S(R)$ around 0 from R to $-R$, for $R > 3$ large enough. Note that $|(z^2+4)(z^2+9)| \geq (R^2-4)(R^2-9)$ when $|z| = R$, so $|\int_{S(R)} f(z)dz| \leq \pi R/((R^2-4)(R^2-9))$, so $\int_{S(R)} f(z)dz \to 0$ as $R \to \infty$. Next, the poles of f are at the points $z = \pm 2i$ and $z = \pm 3i$; the poles in the upper half plane are at $2i$ and $3i$, inside the contour. The residues are easily computed:

$$\mathrm{Res}_{2i}(f) = \frac{1}{4i(-4+9)} = \frac{1}{20i} \qquad \text{and} \qquad \mathrm{Res}_{3i}(f) = \frac{1}{6i(-9+4)} = -\frac{1}{30i}.$$

It follows that $\int_{-\infty}^\infty f(z)\,dz = \frac{2\pi i}{60i}$, so $\int_0^\infty f(z)\,dz = \frac{\pi}{60}$.

Exercise 2 (*15 pt*):

Fix $R > 0$ and $a \in \mathbb{C}$; we write $D := D(a,R)$ and $\overline{D} := \overline{D}(a,R)$ and $C := C(a,R)$. Let $f, g : \overline{D} \to \mathbb{C}$ be continuous functions, analytic on D, such that $|f(z)| = |g(z)|$ for all $z \in C$, and such that f and g have no zeros in $\overline{D}$. Show that $f = \alpha g$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

Solution. Because f and g have no zeros, we have continuous functions $h_1 := f/g$ and $h_2 := g/f$ on $\overline{D}$, analytic on D, such that $|h_1(z)| = |h_2(z)| = 1$ for all $z \in C$. If h_1 is not constant, then h_2 is not constant, and the maximum modulus principle (in the version of Corollary III.1.4) tells us that $|h_1(z)| < 1$ and $|h_2(z)| < 1$ for all $z \in D$, so $|f(z)| < |g(z)|$ and $|g(z)| < |f(z)|$, a contradiction. Hence, $f/g = \alpha$ for some $\alpha \in \mathbb{C}$, and taking $z \in \overline{D}$, we find $|\alpha| = |f(z)/g(z)| = 1$.

Exercise 3 (*15 pt*):

Let $a, b \in \mathbb{C}$. Consider the polynomial $p(z) = z^7 + az^4 + bz^2 - 2$.

(a) Show that if $|z| \le 1/\sqrt{2}$, then

$$|p(z)| \ge \frac{32 - \sqrt{2} - 4|a| - 8|b|}{16}.$$

(b) Suppose that

$$|b| + 3 < |a| \le \tfrac{15}{2} - 2|b|. \tag{1}$$

Show that, counting zeros with their multiplicities, p has

(i) no zeros in the disk $|z| \le 1/\sqrt{2}$,
(ii) four zeros in the annulus $1/\sqrt{2} < |z| < 1$,
(iii) three zeros in the annulus $1 < |z| < 2$,
(iv) and no zeros in the annulus $2 \le |z|$.

Solution **(a)** For $|z| \le 1/\sqrt{2}$ we have, by the (reverse) triangle inequality,

$$\begin{aligned} |p(z)| &\ge 2 - |z^7 + az^4 + bz^2| \\ &\ge 2 - |z|^7 - |a||z|^4 - |b||z|^2 \\ &\ge 2 - \frac{\sqrt{2}}{16} - \frac{|a|}{4} - \frac{|b|}{2} = \frac{32 - \sqrt{2} - 4|a| - 8|b|}{16}. \end{aligned}$$

(b) By the second inequality in (1), we have

$$4|a| + 8|b| \le 30, \tag{2}$$

hence by (a), for $|z| \le 1/\sqrt{2}$ we have $|p(z)| \ge (2 - \sqrt{2})/16 > 0$, which proves (i). For $|z| = 1$, we have

$$|p(z) - az^4| \le |z|^7 + |b||z|^2 + 2 = |b| + 3 < |a| = |az^4|,$$

where in the last inequality we used (1). Hence, by Rouché's Theorem, p has 4 zeros in $D(0,1)$, because az^4 has 4 zeros in that disk. Combining this with (i) yields (ii). Note also that p has no zeros on the circle $|z| = 1$ (we would get $|az^4| < |az^4|$ in the above calculation). For $|z| = 2$, we have, using (2),

$$\begin{aligned} |p(z) - z^7| &\le |a||z|^4 + |b||z|^2 + 2 \\ &= 16|a| + 4|b| + 2 \\ &\le 4(4|a| + 8|b|) + 2 \\ &\le 4 \cdot 30 + 2 < 128 = |z^7|. \end{aligned}$$

Hence, by Rouché's Theorem, p has 7 zeros in $D(0,2)$, because z^7 has 7 zeros in that disk. Combining this with what we found before, namely that p has exactly four zeros in $\overline{D}(0,1)$, yields (iii). Because the number of zeros of a polynomial is at most (even exactly) its degree, there are no other zeros than the seven found in (ii) and (iii), proving (iv).

Exercise 4 (*15 pt*): Let

$$f(z) = \frac{z^2(z-1)e^z}{\sin^2 \pi z}$$

and let $U \subset \mathbb{C}$ be the domain of f. Let $V \subset \mathbb{C}$ be the maximal open set on which a holomorphic function g can be defined that agrees with f on U. For each $v \in V$, determine the radius of convergence of the power series for g at v.

Solution. The numerator and denominator of f are entire functions, so U equals the complement of the zero set of $\sin \pi z$. It's easy to show that $\sin z$ has no zeros in the upper and lower half planes, so $U = \mathbb{C} - \mathbb{Z}$. For each $k \in \mathbb{Z}$, the denominator of f has a double zero. If the numerator has at least a double zero at an integer k, V is larger than U, since the isolated singularity of f at k is then removable. Since e^z never vanishes, this happens only at $k = 0$, so $V = U \cup \{0\}$. From the remark on p. 129 of Lang's book, we see that the radius of convergence of the power series for g at a point v of V is equal to the distance from v to the complement $\mathbb{Z} - \{0\}$ of V. Write $v = a + bi$ with $a, b \in \mathbb{R}$; the answer for $a + bi$ clearly equals the answer for $-a + bi$, so it suffices to give the answer for $a \geq 0$. When $0 \leq a \leq \frac{3}{2}$, the distance equals $\sqrt{(a-1)^2 + b^2}$; when $n - \frac{1}{2} \leq a \leq n + \frac{1}{2}$ for $n \in \mathbb{Z}$ and $n \geq 2$, the distance equals $\sqrt{(a-n)^2 + b^2}$.

Exercise 5 (*15 pt*):
Let f be a non-constant entire function. Prove that the closure of $f(\mathbb{C})$ equals $\mathbb{C}$.

Solution. Suppose that the closure X of $f(\mathbb{C})$ doesn't equal $\mathbb{C}$; let α be an element of $Y = \mathbb{C} - X$. Since Y is open, there exists $\epsilon > 0$ so that $D(\alpha, \epsilon) \subset Y$. Then $|f(z) - \alpha| \geq \epsilon$ for all $z \in \mathbb{C}$. Write $g(z) = 1/(f(z) - \alpha)$. Then g is entire. Also, $|g(z)| \leq 1/\epsilon$ for all $z \in \mathbb{C}$, so g is bounded. By Liouville's theorem, g is constant. Then f is constant as well; contradiction. So $X = \mathbb{C}$.

Exercise 6 (*20 pt*):
Prove that the following integral converges and evaluate it.

$$\int_0^\infty \frac{\log x}{x^3 + 1}\, dx.$$

(*Hint: Use a contour consisting of two circular arcs and two segments, with 'vertices' ϵ, R, Rc, and ϵc, where $c^3 = 1$, $c \neq 1$. Use the natural substitution to relate the integrals over the two segments. Use an appropriate definition of the complex logarithm.*)

Solution. Near zero, the integrand is bounded in absolute value by $-\log x$ and

$$\int_0^1 -\log x\, dx = (x - x\log x)\Big|_0^1 = 1,$$

since $\lim_{x\to 0} x\log x = 0$, so the integral converges near zero. Convergence near infinity follows from an estimate like $|(\log x)/(x^3+1)| < 1/x^2$ for $x > 1$. Let c be the third root of unity in the upper half plane (so $c = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$). Following the hint, we take a contour consisting of the counterclockwise circular arc $A(R)$ around 0 from R to Rc, for $R > 1$ large enough, the clockwise circular arc $-A(\epsilon)$ around 0 from ϵc to ϵ, for $0 < \epsilon < 1/2$ small enough, and the segments from Rc to ϵc and from ϵ to R. Let $f(z) = (\log z)/(z^3+1)$, where we take the (branch of the) complex logarithm on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ that continues $\log x$ for $x > 0$, i.e., for $z = r\exp(i\phi)$ with $r > 0$ and $-\pi < \phi < \pi$ we have $\log z = \log r + i\phi$. Then $\int_{Rc}^{\epsilon c} f(y)\, dy = [y = cx,\ dy = cdx,\ \log y = \log x + 2\pi i/3] = c\int_R^{\epsilon} (\log x + 2\pi i/3)/(x^3+1)\, dx$. Now

$$\int_{A(R)} f(z)\, dz = \int_0^{2\pi/3} \frac{\log R + i\phi}{R^3\exp(3i\phi)+1} iR\exp(i\phi)\, d\phi,$$

so

$$|\int_{A(R)} f(z)\, dz| \leq \frac{2\pi R}{3}\frac{\log R + 2\pi/3}{R^3 - 1},$$

which goes to 0 when $R \to \infty$. Similarly,

$$\int_{A(\epsilon)} f(z)\, dz = \int_0^{2\pi/3} \frac{\log\epsilon + i\phi}{\epsilon^3\exp(3i\phi)+1} i\epsilon\exp(i\phi)\, d\phi,$$

so

$$|\int_{A(\epsilon)} f(z)\, dz| \leq \frac{2\pi\epsilon}{3}\frac{8}{7}(|\log\epsilon| + 2\pi/3),$$

which again goes to 0 as $\epsilon \to 0$.
The integrand has simple poles at the three zeroes of z^3+1. Put $\alpha = \exp(i\pi/3)$, then the only pole in the upper half plane is at α, inside the contour. Now $\mathrm{Res}_\alpha(f) = (\log\alpha)\lim_{z\to\alpha}\frac{z-\alpha}{z^3+1} = \frac{i\pi}{3}\frac{1}{3\alpha^2} = \frac{i\pi}{9c} = \frac{i\pi c^2}{9}$.
So we find that $(1-c)\int_0^\infty f(x)\ dx - 2\pi ic/3\int_0^\infty 1/(x^3+1)\ dx = 2\pi i\frac{i\pi c^2}{9}$.
Hence, we need to compute $\int_0^\infty 1/(x^3+1)\ dx$ (which clearly converges). We take the same contour (in fact, $\epsilon = 0$ suffices now). We find $(1-c)\int_0^\infty 1/(x^3+1)\ dx = \frac{2\pi i}{3\alpha^2} = \frac{2\pi i}{3c}$, so $\int_0^\infty 1/(x^3+1)\ dx = \frac{2\pi i}{3c(1-c)} = \frac{2\pi i}{3i\sqrt{3}} = \frac{2\pi\sqrt{3}}{9}$.
Finally, $\int_0^\infty f(x)\ dx = -\frac{2\pi^2c^2}{9(1-c)} + \frac{2\pi ic}{3(1-c)}\frac{2\pi\sqrt{3}}{9} = \frac{2\pi^2}{27(1-c)}(-3c^2 + 2ic\sqrt{3}) = \frac{2\pi^2}{27}\frac{-\frac{3}{2}+\frac{1}{2}i\sqrt{3}}{\frac{3}{2}-\frac{1}{2}i\sqrt{3}} = -\frac{2\pi^2}{27}$.