ENDTERM COMPLEX FUNCTIONS

JUNE 27 2012, 9:00-12:00

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.

Exercise 1 (*7 pt*) Compute

$$\sum_{n=0}^{\infty} \frac{\sin(nt)}{n!} \qquad (t \in \mathbb{R})$$

Hint: Rewrite the series using the exponential function.

Exercise 2 (*20 pt*) Prove that the following integrals converge and evaluate them.

$$\textbf{a. } (\textit{10 pt}) \int_0^{\infty} \frac{1}{(x^2+i)^2}\,dx \qquad \textbf{b. } (\textit{10 pt}) \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx$$

Exercise 3 (*10 pt*) Let f be an entire function satisfying $|f(-z)| < |f(z)|$ for all z in the upper halfplane ($\mathrm{Im}(z) > 0$).

a. (*7 pt*) Prove that $g(z) = f(z) + f(-z)$ can only have real roots.

b. (*3 pt*) Prove that $z\sin(z) = \cos(z)$ only has real solutions.

Exercise 4 (*8 pt*) Is there an analytic isomorphism between the open unit disc D and $\mathbb{C} \setminus \{a\}$ with $a \in \mathbb{C}$?

Bonus exercise (*15 pt*) Let $f : \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x = 1\} \to \mathbb{C}$ be the sum of $(\log z)^{-2}$ along all the branches of the logarithm, i.e.

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(\log(z) + 2\pi i n)^2}$$

a. (*5 pt*) Prove that f is meromorphic on $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$.

b. (*5 pt*) Prove that f can be analytically continued to $\mathbb{C} \setminus \{1\}$.

c. (*5 pt*) Prove this analytic continuation is a rational function.