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ENDTERM COMPLEX FUNCTIONS

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Exercise 1 (*7 pt*) Compute

$$\sum_{n=0}^{\infty} \frac{\sin(nt)}{n!} \qquad (t \in \mathbb{R})$$

Hint: Rewrite the series using the exponential function.

We know that the analytic function $e^z - e^{1/z}$ has a Laurent expansion in $z = 0$ which converges for $|z| > 0$. By using the exponential series we find

$$e^z - e^{1/z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{z^n - z^{-n}}{n!}$$

By substituting $z = e^{it}$ and using the identity $e^{it} - e^{-it} = 2i \sin t$ we find

$$\sum_{n=0}^{\infty} \frac{\sin(nt)}{n!} = \frac{1}{2i}\left(e^{e^{it}} - e^{e^{-it}}\right) = \frac{1}{2i}\left(e^{\cos t + i \sin t} - e^{\cos t - i \sin t}\right) = e^{\cos t} \sin\sin t.$$

Remark: one could also use

$$\sum_{n=0}^{\infty} \frac{\sin(nt)}{n!} = \mathrm{Im}\left(\sum_{n=0}^{\infty} \frac{e^{int}}{n!}\right) = \mathrm{Im}\left(e^{e^{it}}\right) = e^{\cos t} \sin\sin t.$$

Exercise 2 (*20 pt*) Prove that the following integrals converge and evaluate them.

$$\textbf{a.}\ (\textit{10 pt})\ \int_0^{\infty} \frac{1}{(x^2+i)^2}\, dx \qquad \textbf{b.}\ (\textit{10 pt})\ \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\, dx$$

a. Our integration contour is a line segment L_R from $-R$ to R (with $R > 1$) and a semicircle C_R from R to $-R$ in the upper half-plane. The poles of the integrand f are $\pm(1-i)/\sqrt{2}$. Only $(-1+i)/\sqrt{2} = e^{3\pi i/4}$ is inside our integration contour. We notice

$$\left|\int_{C_R} f(z)dz\right| \leq \int_0^{\pi} \left|f(Re^{it}) iRe^{it}\right| dt \leq \frac{\pi R}{(R^2-1)^2} \to 0 \text{ as } R \to \infty.$$

Where we applied the reverse triangle inequality to the denominator. We conclude that

$$\begin{aligned}\int_0^\infty \frac{dx}{(x^2+i)^2} &= \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+i)^2} = \pi i \operatorname{Res}(f, (-1+i)/\sqrt{2}) \\ &= \pi i \lim_{z\to(-1+i)/\sqrt{2}} \frac{d}{dz}(z-(-1+i)/\sqrt{2})^2 f(z) \\ &= \pi i \lim_{z\to(-1+i)/\sqrt{2}} -2/(z+(-1+i)/\sqrt{2})^3 = -\frac{2\pi i}{8e^{9\pi/4}} = -\frac{\pi}{4}\frac{1+i}{\sqrt{2}}.\end{aligned}$$

Convergence follows from the fact that the integrand is an even function, or that it is asymptotic with $1/x^4$.

b. We will integrate the meromorphic function $f(z) = (1-e^{iz})/z^2$. The integration contour will be a small semicircle C_ϵ from $-\epsilon$ to ϵ in the upper half-plane, a line segment $L_{1,\epsilon,R}$ from ϵ to R, a semicircle C_R from R to $-R$ in the upper half-plane and a line segment $L_{2,\epsilon,R}$ from $-R$ to $-\epsilon$, where $0<\epsilon<R$. We notice

$$\left|\int_{C_R} f(z)dz\right| \leq \int_0^\pi \left|f(Re^{it})iRe^{it}\right| dt \leq \frac{\pi(1+1)R}{R^2} \to 0 \text{ as } R\to\infty.$$

Where we have used that $|e^{iz}| \leq 1$ in the upper half-plane. We know that the pole in $z=0$ is simple and we should take its residue into account by a factor $\frac{1}{2}$ as $\epsilon\to 0$. We conclude that

$$\begin{aligned}\int_{-\infty}^\infty \frac{1-\cos x}{x^2}dx &= \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon}\frac{1-\cos x}{x^2}dx = \lim_{\epsilon\to 0}\lim_{R\to\infty} \operatorname{Re}\left(\int_{L_{1,\epsilon,R}} f(z)dz + \int_{L_{2,\epsilon,R}} f(z)dz\right) \\ &= -\lim_{\epsilon\to 0}\int_{C_\epsilon} f = \pi i \operatorname{Res}(f,0) = \pi.\end{aligned}$$

The first step was allowed because the integrand can be extended to a continuous function (in $x=0$). The integral is convergent because the integrand is an even function. Alternatively one could notice that the integrand is in absolute value less then or equal to $2/x^2$.

Exercise 3 (*10 pt*) Let f be an entire function satisfying $|f(-z)| < |f(z)|$ for all z in the upper halfplane ($\operatorname{Im}(z) > 0$).

a. (*7 pt*) Prove that $g(z) = f(z) + f(-z)$ can only have real roots.

Denote by H the upper half-plane.
The fact that $|f(-z)| < |f(z)|$ for all $z \in H$ implies that f has no roots in H. Now suppose that $g(z_0) = 0$ for some $z_0 \in H$. Consider the path C_0 which is a small circle in H around z_0. On C_0 we have $|f(z) - g(z)| = |f(-z)| < |f(z)|$. Thus by Rouché's theorem f and g should have the same amount of roots inside C_0, which

is a contradiction. We conclude that g has no roots in H. Because g is an even function it also has no roots in $\overline{H}$.

b. (*3 pt*) Prove that $z\sin(z) = \cos(z)$ only has real solutions.

The equation is equivalent to

$$0 = 2i(z\sin z - \cos z) = ze^{iz} - ze^{-iz} - ie^{iz} - ie^{-iz} = (z-i)e^{iz} + (-z-i)e^{-iz}.$$

Denote $f(z) = -(z+i)e^{-iz}$. We notice that for all $z \in H$

$$|f(-z)| = |-(-z+i)e^{-i(-z)}| = |z-i|e^{-\mathrm{Im}(z)} < |z-(-i)|e^{\mathrm{Im}(z)} = |f(z)|.$$

Thus it follows directly from a. that all solutions are real.

Exercise 4 (*8 pt*) Is there an analytic isomorphism between the open unit disc D and $\mathbb{C} \setminus \{a\}$ with $a \in \mathbb{C}$?

No. Suppose there would be such an analytic isomorphism $f : \mathbb{C} \setminus a \to D$. Then f is bounded in a neighborhood of a and thus f can be extended to an entire function $\tilde{f}$. But then $|\tilde{f}| \leq 1 + |\tilde{f}(a)|$. By Liouville's theorem $\tilde{f}$, and thus f, is constant; a contradiction.

Remark: one can also notice that D has a trivial fundamental group while $\mathbb{C} \setminus \{a\}$ has fundamental group (isomorphic to) $\mathbb{Z}$. An analytic isomorphism is in particular a homeomorphism and since the fundamental group is a topological invariant we have reached a contradiction.

Bonus exercise (*15 pt*) Let $f : \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x = 1\} \to \mathbb{C}$ be the sum of $(\log z)^{-2}$ along all the branches of the logarithm, i.e.

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(\log(z) + 2\pi in)^2}$$

a. (*5 pt*) Prove that f is meromorphic on $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$.

First let us prove that the series actually converges to an analytic function. Let K be a compact set in the domain of f. We choose the argument of the log between $-\pi$ and π (though it is irrelevant for the series). We notice that for all $n \neq 0$

$$|\log(z) + 2\pi in|^2 = (\log|z|)^2 + (2\pi in - \arg(z))^2 \geq \pi^2(2|n|-1)^2.$$

Hence we find for all $z \in K$

$$\left|\sum_{n=\infty}^{\infty} \frac{1}{(\log(z) + 2\pi in)^2}\right| \leq \max_K \left|\frac{1}{\log(z)}\right| + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} < \infty.$$

We conclude that the series defines an analytic function. We notice:

$$\lim_{z\to 1}(z-1)^2 f(z) = \lim_{z\to 1}\left(\frac{z-1}{\log(z)}\right)^2 + 0 = 1.$$

Thus f has a pole of order 2 in $z=1$ and we conclude that f is meromorphic on $\mathbb{C}\setminus\{x\in\mathbb{R}|x\le 0\}$.

b. (*5 pt*) Prove that f can be analytically continued to $\mathbb{C}\setminus\{1\}$.

First we only look at z in $U=\{z\in\mathbb{C}|\mathrm{Re}(z)<0 \text{ and } \mathrm{Im}(z)\neq 0\}$. Define LOG to be the logarithm with the argument between 0 and 2π. Notice that by construction we have on U:

$$\sum_{n=\infty}^{\infty}\frac{1}{(\log(z)+2\pi in)^2} = \sum_{n=\infty}^{\infty}\frac{1}{(\mathrm{LOG}(z)+2\pi in)^2}.$$

However, the right hand side can easily be analytically continued to $\{z\in\mathbb{C}|\mathrm{Re}(z)<0\}$, because it is the composition of two analytic functions on that set (namely the LOG and the series one gets by replacing $\log(z)$ by z in the series for f). We conclude that f can be analytically continued to $\mathbb{C}\setminus\{0,1\}$. We notice that

$$\lim_{z\to 0} f(z) = \sum_{n=\infty}^{\infty}\lim_{z\to 0}\frac{1}{(\log(z)+2\pi in)^2} = 0.$$

Here we could swap the order of the sum and limit due to uniform convergence and the fact that the terms of the series can be extended to continuous functions. We conclude that f is bounded in a neighborhood of 0, hence it can be analytically continued to $\mathbb{C}\setminus\{1\}$.

c. (*5 pt*) Prove this analytic continuation is a rational function.

Define $g(z)=(z-1)^2 f(z)$ for $z\neq 1$ and $g(1)=1$. It follows from a. that this definition makes g entire, so we can write $g(z)=a_0+a_1z+a_2z^2+\ldots$. Notice that $g(1/z)=(1/z-1)^2 f(z)=g(z)/z^2$. Hence

$$\ldots + a_2z^{-2}+a_1z^{-1}+a_0 = a_0z^{-2}+a_1z^{-1}+a_2+\ldots.$$

From this we deduce that $a_n=0$ for all $n>2$. We must conclude that

$$f(z)=\frac{a_0(1+z^2)+a_1z}{(z-1)^2}.$$

i.e. f is a rational function. (One easily shows that $a_0=0$ and $a_1=1$.)