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SOLUTIONS ENDTERM COMPLEX FUNCTIONS

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Exercise 1 (*15 pt*):
Prove that the following integral converges and evaluate it.

$$\int_0^\infty \frac{\cos(\frac{\pi}{2}x)}{x^2-1}\,dx.$$

(*Hint: Use a contour consisting of three semicircles and three segments.*)

Solution. The integrand extends continuously to $x = 1$. For $x > 2$, the integrand is bounded in absolute value by $2/x^2$, so the integral converges. In order to use the residue method, we consider $f(z) = \exp(\pi i z/2)/(z^2-1)$, with simple poles at 1 and -1. Note that $\int_{-b}^{-a} f(z)\,dz = [z = -x,\ dz = -dx] = \int_a^b \exp(-\pi i x/2)/(x^2-1)\,dx$, so $\int_{-b}^{-a} f(z)\,dz + \int_a^b f(z)\,dz = 2\int_a^b \cos(\pi x/2)/(x^2-1)\,dx$ for $b > a > 1$ and also for $0 = a < b < 1$. We integrate $f(z)$ therefore over a contour in the closed upper half plane consisting of a semicircle S_R of radius $R > 2$ around 0, semicircles S_ε and T_ε of radius ε with $0 < \varepsilon < 1$ around -1 and $+1$ respectively, and segments from $-R$ to $-1-\varepsilon$, from $-1+\varepsilon$ to $1-\varepsilon$, and from $1+\varepsilon$ to R. The integral is zero. Let $R \to \infty$ and $\varepsilon \to 0$. The integral over S_R is bounded in absolute value by $2\pi R/R^2$ and goes to zero. Let us give S_ε and T_ε the clockwise orientation. When $\varepsilon \to 0$, the integral over S_ε approaches $-\pi i \mathrm{Res}_{-1} f$ and that over T_ε approaches $-\pi i \mathrm{Res}_1 f$. Their sum goes to $-\pi i(e^{\pi i/2} - e^{-\pi i/2})/2 = \pi \sin(\pi/2) = \pi$. As mentioned, the sum of the integrals over the segments goes to twice the desired integral. We conclude that

$$\int_0^\infty \frac{\cos(\frac{\pi}{2}x)}{x^2-1}\,dx = -\frac{\pi}{2}\,.$$

Exercise 2 (*15 pt*):
Determine the fractional linear transformations F that map $\mathbb{R}$ to $\mathbb{R}$ and the unit circle to the unit circle. (As you know, the domain of F equals either $\mathbb{C}$ or the complement of exactly one point. The precise meaning of the above is that F maps the real points in its domain to $\mathbb{R}$, and the points on the unit circle in its domain to the unit circle.)

Solution. We know that there exist a, b, c, and $d \in \mathbb{C}$ with $ad - bc \neq 0$ such that $F(z) = (az+b)/(cz+d)$ for all z in the domain. When $c = 0$, the domain is $\mathbb{C}$, otherwise it is $\mathbb{C} \setminus \{-d/c\}$. Analogously, the image is either $\mathbb{C}$ or $\mathbb{C} \setminus \{a/c\}$. We also know that F gives a bijection from its domain to its image. In case $c \neq 0$: when $z \to -d/c$, then $F(z) \to \infty$. It follows that the domain of F contains the unit circle. In particular, F is defined at 1 and -1. Moreover, F preserves $\{1, -1\}$, the intersection of the real line and the unit circle. Let $z \to \infty$ along the real line; it follows that $c = 0$ or a/c is real.
Assume first that $F(1) = 1$ and $F(-1) = -1$. Then $a + b = c + d$ and $-a + b = c - d$, so $b = c$ and $a = d$ and $a^2 \neq b^2$. If $a = 0$, then $F(z) = 1/z$, preserving the unit circle and $\mathbb{R} \setminus \{0\}$. Otherwise, we can take $a = 1$ and $F(z) = (z + b)/(bz + 1)$, with b real. Clearly, F maps the real points in its domain to real points. When $|z| = 1$, then $|bz + 1| = |b + 1/z| = |b + \overline{z}| = |b + z|$, so $|F(z)| = 1$ and F preserves the unit circle.
Next, if $F(1) = -1$ and $F(1) = -1$, then $F(-z)$ (or $-F(z)$) gives a fractional linear transformation that fixes 1 and -1, preserves the unit circle and maps real points to real points. So it is of the form above. Conclusion: $F(z) = \pm(1/z)$ or $F(z) = \pm(z+b)/(bz+1)$ for b real, $b^2 \neq 1$, are precisely the fractional linear transformations asked for.

Exercise 3 (*15 pt*):
Let $U \subseteq \mathbb{C}$ be a nonempty open and connected set. A function $f\colon U \to \mathbb{C}$ is *distance preserving* when $|f(z) - f(w)| = |z - w|$ for all z and w in U. Determine (with proof) the distance preserving holomorphic functions on U. (State the theorems (or their names) that you use. As mentioned above: when you use a theorem, show that the conditions are met.)

Solution. Assume that $f\colon U \to \mathbb{C}$ is holomorphic and distance preserving. We know that the derivative of f exists. Also, $f'(z) = \lim_{w \to z} \frac{f(z)-f(w)}{z-w}$, so we find that $|f'(z)| = 1$ for all $z \in U$. Since f is analytic on U, we have that f' is analytic as well. But its image is contained in the unit circle, so the open mapping theorem tells us that f' is locally constant, hence constant, since U is connected. So f' is constant, of absolute value 1, hence there exist a and b in $\mathbb{C}$ with $|a| = 1$ such that $f(z) = az + b$ for all $z \in U$ (again using the connectedness: if $f' \equiv a$, then $f(z) - az$ has zero derivative, hence is constant on U). Since f is a rotation followed by a translation, it is distance preserving.

Exercise 4 (*15 pt*):
Let $S \subset \mathbb{C}$ be a closed set that is discrete (i.e., every point of S is isolated). Let $U \subseteq \mathbb{C}$ be the complement of S. Prove that a holomorphic function f from U to the upper half plane H is necessarily constant.

Solution. We know that H and the open unit disc D are analytically isomorphic; let $g\colon H \to D$ be an analytic isomorphism. Then $g \circ f\colon U \to D$ is analytic and bounded. At each point s of S, the function $g \circ f$ has an isolated singularity, but since $g \circ f$ is bounded in a punctured neighbourhood of s, the singularity is a removable one. So $g \circ f$ extends to a holomorphic function h from $\mathbb{C}$ to $\overline{D}$. Then h is entire and bounded, so constant by Liouville's theorem. Then $g \circ f$ and f are constant as well (with image one point in D resp. H).

Exercise 5 (*15 pt*):
Let f be an entire function that is not a polynomial. Show that for every $c \in \mathbb{C}$ there exists an unbounded sequence $(z_n)_{n\in\mathbb{N}}$ such that $f(z_n) \to c$ as $n \to \infty$.

Solution. As f is entire, we can write f as

$$f(z) = \sum a_n z^n.$$

We set $g(z) := f\left(\frac{1}{z}\right)$ and, because f is not a polynomial, we observe that g has an essential singularity at 0. Pick a point $c \in \mathbb{C}$. Then by the Casorati-Weierstrass Theorem, given an $\varepsilon > 0$, we can find in any punctured neighbourhood V of 0 a point $w \in V$ such that $|g(w) - c| < \varepsilon$. We use this fact to construct our sequence.
Let $n \in \mathbb{N}$ and pick a point $w_n \in D\left(0, \frac{1}{n}\right)$ (with $w_n \neq 0$) such that $|g(w_n) - c| < \frac{1}{n}$. Then we define $z_n = \frac{1}{w_n}$, so that $f(z_n) \to c$ as $n \to \infty$ by construction; clearly, $(z_n)_{n\in\mathbb{N}}$ is unbounded.

Exercise 6 (*15 pt*):
Prove that the following integral converges and evaluate it.

$$\int_0^\infty \frac{(\log x)^2}{x^2+1}\, dx.$$

(*Hint: Use a contour consisting of two semicircles and two segments. Use an appropriate definition of the complex logarithm.*)

Solution. Convergence near ∞ is essentially obvious, e.g., since $x^{-3/2}$ bounds the integrand. Convergence near 0 follows from partial integration, since the limits of $x(\log x)^2$ and $x \log x$ as $x \to 0$ exist (and are 0). Let $f(z) = (\log z)^2/(z^2+1)$, where we take the (branch of the) complex logarithm on $\mathbb{C} \setminus i\mathbb{R}_{\leq 0}$ that continues $\log x$ for $x > 0$, i.e., for $z = r\exp(i\phi)$ with $r > 0$ and $-\pi/2 < \phi < 3\pi/2$ we have $\log z = \log r + i\phi$. Take a contour in the closed upper half plane consisting of the semicircle S_R around 0 of radius $R > 2$, the semicircle S_ε around 0 of radius ε with $0 < \varepsilon < 1/2$, and segments from $-R$ to $-\varepsilon$ and from ε to R. The only pole of f inside the contour is at $z = i$; it is simple, with residue equal to $-(\pi^2/4)/(2i)$ (since $\log i = i\pi/2$), so the integral of f over the contour equals $-\pi^3/4$. Note that $\int_{-R}^{-\varepsilon} f(z)dz = [z = -x,\ dz = -dx,\ \log z = \log x + i\pi] = \int_\varepsilon^R (\log x + i\pi)^2/(x^2+1)\, dx$. Let $R \to \infty$ and $\varepsilon \to 0$. The integral over S_R is bounded in absolute value by $\pi R(\log R + \pi)^2/(R^2-1)$ and goes to 0 as $R \to \infty$. The integral over S_ε is bounded in absolute value by $\pi\varepsilon(|\log\varepsilon| + \pi)^2/(3/4)$ and also goes to 0 as $\varepsilon \to 0$. Taking the real part, we find that

$$2\int_0^\infty \frac{(\log x)^2}{x^2+1}\, dx - \pi^2 \int_0^\infty \frac{dx}{x^2+1} = -\frac{\pi^3}{4}.$$

But we know that $\int_0^\infty \frac{dx}{x^2+1} = \pi/2$, so we find the answer

$$\int_0^\infty \frac{(\log x)^2}{x^2+1}\, dx = \frac{\pi^3}{8}\,.$$

(Taking the imaginary part, we find that $\int_0^\infty \frac{\log x}{x^2+1}\, dx = 0$; we can also obtain this by integration over the same contour, or by a simple substitution. Needless to say, $\int_0^\infty \frac{dx}{x^2+1} = \pi/2$ can also be obtained by contour integration.)