RETAKE COMPLEX FUNCTIONS

JULY 19, 2018, 13:30-16:30

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.
- The use of books, notes, computers, calculators, mobile phones, etc., is not allowed.

Exercise 1 (*15 pt*):

a. (*5 pt*) Find the four complex roots of $z^4 + z^2 + 1 = 0$. (*Remark: the answer will be useful to you in Exercise 2 below.*)

b. (*5 pt*) State a version (or several versions) of the **maximum modulus principle**.

c. (*5 pt*) Let U be a nonempty, open, and connected set in $\mathbb{C}$. Determine the holomorphic functions f on U such that $|f'''(z)| = 2$ for $z \in U$.

Exercise 2 (*20 pt*):

a. (*10 pt*) The following integral clearly converges. Evaluate it.

$$\int_0^\infty \frac{1}{x^4 + x^2 + 1}\, dx.$$

b. (*10 pt*) The following integral converges. Evaluate it.

$$\int_0^\infty \frac{\log(x)}{x^4 + x^2 + 1}\, dx.$$

(*Hint: Use a contour consisting of two semicircles and two segments. Use an appropriate definition of the complex logarithm.*)

Please turn over.

Exercise 3 (*40 pt*):
The goal of this exercise is to prove that the following identity holds:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 \sinh(\pi n)} = \frac{\pi^3}{360}$$

(Recall that $2\sinh(z) = e^z - e^{-z}$.) The proof is divided in a number of steps. Even if you fail to complete one of the steps, you may be able to complete later ones.
The idea is to consider the integral of

$$f(z) = \frac{\pi}{z^3 \sinh(\pi z)\sin(\pi z)}$$

over the square γ_N with vertices $(\pm(N+\frac{1}{2}), \pm(N+\frac{1}{2})i)$, for $N \in \mathbb{N}$.

a. (*5 pt*) Let T_N be one of the sides of the square γ_N, say the top side. Prove that there exists a positive lower bound, independent of N, for $|\sinh(\pi z)|$ on T_N (i.e., for all $z \in T_N$).

b. (*5 pt*) Prove that there exists a positive lower bound, independent of N, for $|\sin(\pi z)|$ on T_N.

c. (*3 pt*) Prove that such bounds also exist for the other sides of the square.

d. (*5 pt*) Deduce that

$$\int_{\gamma_N} f \to 0 \quad \text{as } N \to \infty.$$

e. (*3 pt*) Determine all the poles of f.

f. (*5 pt*) At each nonzero pole of f, determine the residue of f.

g. (*3 pt*) Prove that the sum of all the residues of f converges absolutely and is equal to zero.

h. (*8 pt*) Determine the residue of f at 0. (This is somewhat difficult.)

i. (*3 pt*) Complete the proof of the desired identity.

Exercise 4 (*15 pt*):
Consider the punctured unit disc $D \setminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Let $S = \{z_n \mid n \in \mathbb{N}\}$ be an infinite subset of $D \setminus \{0\}$ satisfying

$$\lim_{n\to\infty} z_n = 0.$$

Note that each point of S is isolated. Finally, let f be a holomorphic function on the complement U of S in $D \setminus \{0\}$, and assume that f has poles at all the points of S. Show that the image of f is dense in $\mathbb{C}$.