Complex analysis – Mock Exam

Notes:

1. **Write your name and student number *****clearly******* on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books, the lecture's slides and your own notes.
5. You are **not** allowed to consult colleagues, calculators, or use the internet to assist you solve exam questions.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1. Let $a_1, \dots, a_n \in \mathbb{C}$ be a collection of complex numbers of norm 1. Show that there is a point inside the unit disc such that $\Pi_{i=1}^n |z - a_n| > 1$.

Exercise 2. Let f be analytic on a closed disc $\bar{D}$ of radius $b > 0$, centered at z_0.

- Show that the value of f at z_0 can be computed as either of the following two averages:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta, \text{ where } 0 < r < b$$

$$f(z_0) = \frac{1}{\pi b^2} \int_D f(x + iy) dy dx.$$

- Is the converse true? That is, if a continuous function $f\colon U \to \mathbb{C}$ satisfies

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta,$$

for all $z \in U$ and all r such that $\bar{D}_r(z) \subset U$, is f holomorphic?

Exercise 3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be the holomorphic function with singularities given by

$$f(z) = \frac{e^{-2\pi i z}}{z^3 + i}.$$

- Determine the singularities of f and for each of them, determine what type of singularity it is (removable, pole or essential).

- Compute the residue of f at each of its singularities.
- Compute the integrals

$$\int_{-\infty}^{\infty} \frac{x^3 \cos 2\pi x - \sin 2\pi x}{x^6+1} dx.$$

$$\int_{-\infty}^{\infty} \frac{x^3 \sin 2\pi x - \cos 2\pi x}{x^6+1} dx.$$

Exercise 4. Consider the group homomorphism

$$\Phi\colon \mathrm{Sl}(2;\mathbb{C}) \to \text{Möb}, \qquad \Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az+b}{cz+d}.$$

- Show that if $v = (z_1, z_2)$ is an eigenvector of A, that is $Av = \lambda v$ for some $\lambda \in \mathbb{C}$, then $z = \frac{z_1}{z_2}$ is a fixed point for $\Phi(A)$.
- Show that if z is a fixed point for $\Phi(A)$, then $(z, 1)$ is an eigenvector for A.

Exercise 5. Let $u\colon \mathbb{C} \to \mathbb{R}$ be a harmonic function and $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Prove or disprove the following statements.

- $u \circ f$ is harmonic,
- $f \circ u$ is holomorphic.