Complex analysis – Retake Exam

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books, the lecture's slides and your own notes.
5. You are **not** allowed to consult colleagues, calculators, or use the internet to assist you solve exam questions.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Notation

- $D := \{z \in \mathbb{C}\colon |z| < 1\}$ is the unit disc and $D^* = D\backslash\{0\}$ is D with the origin removed.
- $H := \{z \in \mathbb{C}\colon \mathrm{Im}(z) > 0\}$ is the upper half plane.

Questions

Exercise 1. Let f be an entire function. Prove that in each of the following cases, f is constant:

1. (1.0 pt) f satisfies $\mathrm{Im}(f(z)) \leq 0$ for all $z \in \mathbb{C}$.
2. (1.0 pt) f does not receive any value in $\mathbb{R}_- = \{x \in \mathbb{R}\colon x \leq 0\}$.

Exercise 2 (1.5 pt)**.** Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function and assume that there is $N \in \mathbb{N}$ such that $|f(z)| \geq |z|^N$ for sufficiently large $|z|$. Show that f is a polynomial.

Exercise 3 (1.5 pt)**.** Let $f\colon D^* \to \mathbb{C}$ be holomorphic. Show that if f takes no real values, then 0 is a removable singularity

Exercise 4. Let $f\colon D \to D$ be holomorphic and 0 be a zero of f of order $n \geq 1$. Show that

1. (0.7 pt) $|f(z)| \leq |z|^n$ for all $z \in D$.
2. (0.7 pt) $\frac{d^n f}{dz^n}(0) \leq n!$ and
3. (0.6 pt) Show that if there is $z_0 \in D\backslash\{0\}$ such that $|f(z_0)| = |z_0|^n$ then there exists $a \in \mathbb{C}$ with $|a| = 1$ such that $f(z) = az^n$.

Exercise 5 (1.5 pt)**.** Let $z_1, \ldots, z_k \in \mathbb{C}$ be distinct points. Let $f\colon \mathbb{C}\backslash\{z_1, \ldots, z_k\} \to \mathbb{C}$ be a holomorphic function such that $\lim_{|z|\to\infty} f(z) = 0$. Prove that $\lim_{|z|\to\infty} zf(z)$ exists and

$$\sum_i \operatorname{Res}_{z_i}(f) = \lim_{|z|\to\infty} zf(z).$$

Exercise 6 (1.5 pt)**.** Let $a > 1$. Compute the integral

$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2}.$$