Complex analysis – Exam

Notes:

1. **Write your name and student number ****clearly**** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books, the lecture's slides and your own notes.
5. You are **not** allowed to consult colleagues, calculators, or use the internet to assist you solve exam questions.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (1.0 pt)**.** For a real number x, we know that $\sin^2 x + \cos^2 x = 1$. If we extend sin and cos to the complex plane using their respective power series expansions, does it still hold that $\sin^2 z + \cos^2 z = 1$ for all complex numbers?

Exercise 2 (1.0 pt)**.** Let $f, g\colon \mathbb{C} \to \mathbb{C}$ be holomorphic functions such that $|f(z)| \leq |g(z)|$ for all z. Show that there is a complex number λ such that $f = \lambda g$.

Exercise 3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a function which is bounded by $\log |z|$ for $|z|$ large, that is, there is $C > 1$ such that if $|z| > C$, then $|f(z)| \leq \log |z|$.

1. (1.0 pt) Show that if f is holomorphic, then f is constant.
2. (1.0 pt) If f is harmonic, does it have to be constant?

Exercise 4. Let $f\colon \mathbb{C} \to \mathbb{C}$ be the holomorphic function with singularities given by

$$f(z) = \frac{e^{iz}}{z^4 + 1}.$$

1. (0.7 pt) Determine the singularities of f and for each of them, determine what type of singularity it is (removable, pole or essential).
2. (0.7 pt) Express the residue of f at each of its singularities in terms of the 8^{th} root of 1, $\omega = e^{\frac{2\pi i}{8}}$.
3. (0.6 pt) Relate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1} dx$$

   to the residues of f.

Exercise 5 (1.5 pt)**.** Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by

$$f(z) = \frac{z^4}{1 + z + 2z^2 \cdots + 1000z^{1000}}.$$

Compute the sum of all residues of f.

Exercise 6 (1.5 pt)**.** Let a be a real number bigger that 1. Show that the equation

$$e^z - z^n e^a = 0$$

has n solutions inside the unit disc (counted with multiplicity).

Exercise 7.

1. (0.5) Show that the first quadrant

   $$R = \{(x + iy) \in \mathbb{C} \colon x > 0, \text{ and } y > 0\}$$

   is isomorphic to the upper half plane

   $$H = \{(x + iy) \in \mathbb{C} \colon y > 0\}.$$

2. (0.5) Determine all the automorphisms of R.