TENTAMEN COMPLEX FUNCTIONS

FEBRUARY 4 2005

- You may do this exam either in English or in Dutch.
- Put your name, studentnummer (and email address if you have one) on the first sheet and put your name on every other sheet you hand in.
- Give only reasoned solutions, but try to be concise.

(1) (a) Prove that e^{1/z^n} has an essential singularity at 0 when n is a positive integer.
(b) Let $f \in \mathbb{C}[z]$ be a polynomial in z. Prove that e^f has an essential singularity at ∞ unless f is constant.
(c) Let f be a holomorphic function on all of $\mathbb{C}$ with the property that e^f is a polynomial. Prove that f must be a constant.

(2) Consider the polynomial function $f(z) := z^8 + 2z + 1$.
(a) Determine the number of zeroes of f on $|z| < 1$.
(b) Prove that -1 is the only zero of f on the circle $|z| = 1$.
(c) Prove that f has no zeroes of multiplicity > 1. How many zeroes will f therefore have on $|z| > 1$?

(3) Compute for $0 < s < 1$ the integral

$$\int_0^{2\pi} \frac{dx}{1 + s\cos x}.$$

(4) Prove that the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1}\, dx$$

exists and compute its value.

(5) Give a biholomorphic map from the open unit disk $|z| < 1$ onto the open half disk defined by $|z| < 1, \mathrm{Im}(z) > 0$.

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