Exam Complex functions February 2005—Solutions

1a. *Prove that e^{1/z^n} has an essential singularity at 0 when n is a positive integer.* If not, then e^{1/z^n} is meromorphic at 0 and so $\lim_{z\to 0} e^{1/z^n}$ exists (with ∞ allowed). But if we take $(z = \frac{1}{k})_{k=1}^{\infty}$ we get $\lim_{k\to\infty} e^{k^n} = \infty$, wheras for $(z = \frac{1}{2\pi ik})_{k=1}^{\infty}$ we get $\lim_{k\to\infty} e^{2\pi ik^n} = 1$. So the singularity is essential.

Other proof: for every $\varepsilon > 0$, the function $w = z^n$ maps the punctured disk $0 < |z| < \varepsilon$ onto the punctured disk $0 < |w| < \varepsilon^n$; since $e^{1/w}$ has an essential singularity at 0, this function maps $0 < |w| < \varepsilon^n$ onto a dense subset of $\mathbb{C}$. Hence e^{1/z^n} maps $0 < |z| < \varepsilon$ onto a dense subset of $\mathbb{C}$. This also implies that e^{1/z^n} has an essential singularity at 0.

1b. *Let $f \in \mathbb{C}[z]$ be a polynomial in z. Prove that e^f has an essential singularity at ∞ unless f is constant.*
Suppose f nonconstant. Since f is meromorphic and nonconstant at ∞, we have that for every $R > 0$, $w := f(z)$ sends $|z| > R$ to a punctured neighborhood of ∞, i.e., its image contains a subset of the form $|w| > R'$ for some for some $R' > 0$. But since e^w has an essential singularity at ∞, the image of $|w| > R'$ under e^w is dense in $\mathbb{C}$. So e^f has then an essential singularity at ∞.

2. *Consider the polynomial function $f(z) := z^8 + 2z + 1$.*
2a. *Determine the number of zeroes of f on $|z| < 1$.*
We compare f with $g(z) := 2z + 1$. On $|z| = 1$ we have $|2z + 1| \geq 1$ with equality only when $z = -1$, whereas $|f(z) - g(z)| = |z|^8 = 1$. So on $|z| = 1$, we have $|f(z) - g(z)| \leq |g(z)|$ with equality only if $z = -1$. Since the inequality is not strict, the Rouché principle does not apply for this radius; we therefore take it slightly smaller: $|z| = 1 - \varepsilon$ with $\varepsilon > 0$ very small. Then $|2z + 1| \geq 1 - 2\varepsilon$ and $|z|^8 = (1-\varepsilon)^8 = 1 - 8\varepsilon + o(\varepsilon)$ and so $|g(z)| - |f(z) - g(z)| = 6\varepsilon + o(\varepsilon)$ on $|z| = 1 - \varepsilon$ and hence positive for sufficiently small $\varepsilon > 0$. According to the Rouché principle, f has then in $|z| < 1 - \varepsilon$ as many zeroes (counted with multiplicity) as g. The latter has $z = -\frac{1}{2}$ as its only zero, so this number is one. As we can take ε as small as we please, it follows that f has only one zero in $|z| < 1$.
2b. *Prove that -1 is the only zero of f on the circle $|z| = 1$.*
If z is a zero of f with $|z| = 1$, then $|2z + 1| = |-z^8| = 1$ and this implies $z = -1$.
2c. *Prove that f has no zeroes of multiplicity > 1. How many zeroes will f therefore have on $|z| > 1$?*
If z is a zero of f of order ≥ 2, then $f(z) = f'(z) = 0$, i.e., $z^8 + 2z + 1 = 8z^7 + 2 = 0$. Hence $z^7 = -\frac{1}{4}$ and so $0 = z^8 + 2z + 1 = \frac{7}{4}z + 1$. It follows that $z = -\frac{4}{7}$. But $(-\frac{4}{7})^7 \neq -\frac{1}{4}$ and so such a z does not exist. Hence f has as many zeros as its degree, namely 8. In view of 2a and 2b, this implies that f has exactly 6 zeroes on $|z| > 1$.

3. *Compute for $0 < s < 1$ the integral $\int_0^{2\pi} \frac{dt}{1+s\cos t}$.*
This is a trigonometric integral and so we use the substitution $z := e^{it}$. Then

$dt = -iz^{-1}dz$ and $\cos t = \frac{1}{2}(z+z^{-1})$ so that

$$\int_0^{2\pi} \frac{dt}{1+s\cos t} = -i\int_{|z|=1} \frac{dz}{z(1+\frac{s}{2}(z+z^{-1}))} = -\frac{2i}{s}\int_{|z|=1} \frac{dz}{z^2+\frac{2}{s}z+1}.$$

The denominator $z^2+\frac{2}{s}z+1$ factors as $(z-z_+)(z-z_-)$ with $z_\pm = -s^{-1} \pm \sqrt{s^{-2}-1}$. It has z_+ as its unique zero lying in $|z|<1$ and the residue of $(z^2+\frac{2}{s}z+1)^{-1}$ in this point is $(z_+-z_-)^{-1} = (2.\sqrt{s^{-2}-1})^{-1}$. So the integral we are after is by the residue theorem equal to

$$-\frac{2i}{s}.2\pi i\frac{1}{2.\sqrt{s^{-2}-1}} = \frac{2\pi}{\sqrt{1-s^2}}.$$

4. *Prove that the integral* $\int_{-\infty}^{\infty}\frac{\cos 2x}{x^2+1}\,dx$ *converges and compute its value.*
The integral converges (absolutely) because for $|x|>1$, $|\frac{\cos 2x}{x^2+1}| \le 2|x|^{-2}$ and $\int_1^\infty 2x^{-2}dx < \infty$. In order to compute it, we consider for $R>1$ the integral

$$I(R) := \int_{\Gamma_R}\frac{e^{2iz}}{z^2+1},$$

where Γ_R is the closed path which first traverses the real interval $[-R,R]$ and then the semicircle $\Gamma_R' : t\in[0,\pi]\mapsto Re^{it}$. The integral $I(R)$ is computed by means of the residue formula: we factor the denominator $z^2+1=(z-i)(z+i)$. Its zero inside Γ_R is i and the residue of $(z^2+1)^{-1}e^{2iz}$ at this point is $(2i)^{-1}e^{-2}$. It follows that $I(R) = 2\pi i(2i)^{-1}e^{-2} = \pi e^{-2}$.

For $t\in[0,\pi]$ and $R>1$, we have

$$\left|\frac{e^{2iR(e^{it})}}{e^{2it}+1}\right| = \frac{e^{-2R\,\mathrm{Im}(e^{it})}}{|e^{2it}+1|} \le \frac{e^{-2R\sin t}}{R^2-1} \le \frac{1}{R^2-1}.$$

and so

$$\left|\int_{\Gamma_R'}\frac{e^{iz}}{z^2+1}\right| = \left|\int_0^\pi \frac{e^{(iRe^{it})}}{e^{2it}+1}.iRe^{it}\,dt\right| \le \pi\frac{R}{R^2-1}.$$

The latter goes to zero as $R\to\infty$. It follows that $\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{2ix}}{x^2+1}dx = \pi e^{-2}$. Taking the real part yields $\int_{-\infty}^{\infty}\frac{\cos 2x}{x^2+1}dx = \pi e^{-2}$.

5. *Give a biholomorphic map from the open unit disk onto the open half disk defined by* $|z|<1, \mathrm{Im}(z)>0$.
Recall that $w\mapsto (w-i)(w+i)^{-1}$ maps the upper half plane H_+ onto the open unit disk Δ. So its inverse, $z\mapsto w = -i(z+1)(z-1)^{-1}$, maps Δ biholomorphically onto H_+. The function $w=\frac{1}{2}(\zeta+\zeta^{-1})$ maps the lower half disk Δ_- biholomorphically onto H_+: if $\zeta\in\Delta_-$, then $\mathrm{Im}(w) = \frac{|\zeta|^2-1}{2|\zeta|^2}\mathrm{Im}(\zeta) > 0$ so that $w\in H_+$. The inverse map $H_+\to\Delta_-$ is given by picking the root of $\zeta^2-2\zeta w+1$: one root satisfies $|\zeta|>1$ and $\mathrm{Im}(\zeta)>0$ and the other $|\zeta|<1$ and $\mathrm{Im}(\zeta)<0$. We take the latter and denote it by $\zeta(w)$ (in fact, $\zeta(w) = -w+\sqrt{w^2-1}$, where the square root is taken with its argument in $(0,\pi)$). Then $z\mapsto -\zeta(-i(z+1)(z-1)^{-1})$ is as desired.