Exam Complex Functions

December 19, 2000, 14–17

1. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function so that $\bar{f}$ is also holomorphic. Determine $\frac{d}{dz} f(z)$.

2. Let $\sum_{j=0}^{\infty} a_j z^j$ be convergent on $D(0,1)$. Let $b_j \in \mathbb{C}$ satisfy $|a_j - b_j| \leq 1$ for $j = 0, 1, \ldots$
   Show that the radius of convergence of $\sum_{j=0}^{\infty} b_j z^j$ is at least one.

3. Let $a \in \mathbb{R}, \quad a > 1$. Compute
   $$\int_{-\pi}^{\pi} \frac{\cos(\phi)}{a + \cos(\phi)} \, d\phi$$
   by applying the Residue Theorem.

4. Let $\gamma : [0,1] \to D(0,1), \quad \gamma(0) = \gamma(1)$ be a closed piecewise C^1 curve. Let $0 \leq r \leq 1$. Clearly the map $r\gamma : t \mapsto r\gamma(t)$ is a closed piecewise C^1 curve with values in $D(0,r)$. Let $f : D(0,1) \to \mathbb{C}\backslash\{0\}$ be holomorphic.
   (a) Give at least one proof that
   $$\oint_{r\gamma} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta$$
   vanishes.
   (b) Show that $\frac{f'(z)}{f(z)}$ has a holomorphic antiderivative on $D(0,1)$.

5. Let $r > 0$. Let $f : D(P,r)\backslash\{P\} \to \mathbb{C}$ be a holomorphic function with an essential singularity at P. Show that f is not one-to-one.
   Hint: Combine the Open Mapping Theorem with the Casorati–Weierstrass theorem.