INSTITUTE OF MATHEMATICS, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UU.
MADE AVAILABLE IN ELECTRONIC FORM BY THE $\mathcal{T}_{\!\mathcal{B}}\mathcal{C}$ OF A–ESKWADRAAT
IN 2007/2008, THE COURSE WISB311 WAS GIVEN BY ERIK VAN DEN BAN.

Complex Functions (WISB311)
02 July 2007

Remark: The exam also had a dutch version. Because of time constraints, it is omitted here.

Question 1

Let $\alpha \in \mathbb{C}$, let $r > 0$ and assume that the function f is holomorphic on an open neighborhood in $\mathbb{C}$ of the closed disk $\overline{D}_r := \overline{D}(\alpha; r) : |z - \alpha| \leq r$

a) Show that for all $z_1, z_2 \in D_r := D(\alpha; r)$,

$$f(z_1) - f(z_2) = \frac{z_1 - z_2}{2\pi\, i} \int_{\partial D_r} \frac{f(w)}{(w - z_1)(w - z_2)} dw.$$

Hint: Use Cauchy's integral formula for $f(z_j), j = 1, 2$

b) Assume that $|f(w)| \leq M$ for all $w \in \partial D_r$. Show that

$$|f(z_1) - f(z_2)| \leq \frac{4M}{r} |z_1 - z_2|$$

for all z_1, z_2 in the disk $D_{r/2} := D(\alpha, r/2)$.

Question 2

We consider the polynomial function $p(z) = z^{10} + (4 + 3\,i)z^8 - z^2 + i$

a) Show that $|z| > \frac{1}{2}$ for any zero z of p.

b) Determine the number of zeros of p contained in the annulus $1 < |z| < 2$. (Zeros should be counted with multiplicities.)

c) Same question, but now for the annulus $2 < |z| < 3$.

Question 3

We consider the integral

$$I = \int_0^\infty \frac{1}{1 + x + x^2} \frac{dx}{\sqrt{x}}.$$

To compute it, we first calculate two residues. We denote by $s(z)$ the holomorphic function on $\mathbb{C}\backslash[0, \infty[$ (i.e., $\mathbb{C}$ minus the positive real axis) determined by

$$s(z)^2 = z, \qquad \text{and} \qquad s(-1) = i.$$

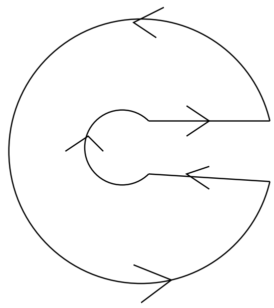
a) Give an expression for $s(z)$ in terms of a suitable logarithmic function.

b) Let α be the unique root of the polynomial $1 + z + z^2$ with $\operatorname{Im} \alpha > 0$ Determine the residue of the function

$$\frac{1}{1 + z + z^2} \frac{1}{s(z)}$$

in α. In addition, determine the residue of this function in the conjugate point $\overline{\alpha}$.

c) Calculate the integral I. Hint: use the following closed curve:



Question 4

Give an invertible complex 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we denote by $F_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the fractional linear transformation given by

$$F_A(z) = \frac{a\,z + b}{c\,z + d}.$$

a) Determine a matrix M such that the associated fractional linear transformation F_M maps the points $1, i, -1$ onto $0, 1, \infty$, respectively.

b) Prove that F_M maps the unit circle $|z| = 1$ onto $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$.

c) Prove that F_M maps the unit disk $D = \{z \in \mathbb{C} |\ |z| < 1\}$ bijectively onto the upper half plane $\mathcal{H} = \{z \in \mathbb{C} |\ \mathrm{Im}\, z > 0\}$.

d) Let F be any fractional linear transformation mapping D onto $\mathcal{H}$. Show that for every point $z \in D$ we have $F(1/\overline{z}) = \overline{z}$. Hint: show first that $z \mapsto \overline{F(1/\overline{z})}$ is analytic, and that the identity holds for $z \in \partial D$. If you fail to see the argument, show at least that identity is valid for $F = F_M$.