Universiteit Utrecht Boedapestlaan 6
Mathematisch Instituut 3584 CD Utrecht

Final Measure and Integration 2012-13

(1) Let $(E, \mathcal{B}, \nu)$ be a measure space, and $h : E \to \mathbb{R}$ a non-negative measurable function. Define a measure μ on $(E, \mathcal{B})$ by $\mu(A) = \int_A h d\nu$ for $A \in \mathcal{B}$. Show that for every non-negative measurable function $F : E \to \mathbb{R}$ one has

$$\int_E F\, d\mu = \int_E F h\, d\nu.$$

Conclude that the result is still true for $F \in \mathcal{L}^1(\mu)$ which is not necessarily non-negative. (Hint: use a standard argument starting with indicator functions) (20 pts)

(2) Consider the measure space $((0,\infty), \mathcal{B}((0,\infty), \lambda)$, where $\mathcal{B}((0,\infty))$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure to the interval $(0,\infty)$. Show that

$$\lim_{n\to\infty} \int_{(0,n)} \left(1 + \frac{x}{n}\right)^n e^{-2x}\, d\lambda(x) = 1.$$

(Hint: note that $1 + x \le e^x$). (20 pts)

(3) Let $(X, \mathcal{A}, \mu)$ be a probability space (i.e. $\mu(X) = 1$) and let $\{f_n\}$ be a sequence in $\mathcal{L}^1(\mu)$ such that $\int_X |f_n| d\mu = n$ for all $n \ge 1$. Let

$$A_n = \{x : |f_n(x) - \int_X f_n d\mu| \ge n^3\}.$$

(a) Show that $\mu\left(\bigcap_{m\ge 1} \bigcup_{n\ge m} A_n\right) = 0$. (Hint: use Exercise 6.9 (Borel-Cantelli Lemma)). (10 pts)

(b) Use part (a) to show that for every $\epsilon > 0$ there exists $m_0 \ge 1$ such that

$$\mu\{x \in X : |f_n(x)| < n^3 + n,\ \text{for all } n \ge m_0\} > 1 - \epsilon.$$

(10 pts)

(4) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and (A_i) a sequence in $\mathcal{A}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$.

(a) Show that $\mathbf{1}_{A_n} \xrightarrow{\mu} 0$, i.e. the sequence $(\mathbf{1}_{A_n})$ converges to 0 in measure. (5 pts)

(b) Show that for any $u \in \mathcal{L}^1(\mu)$, one has $u\mathbf{1}_{A_n} \xrightarrow{\mu} 0$.(5 pts)

(c) Show that for any $u \in \mathcal{L}^1(\mu)$, one has

$$\sup_n \int_{\{|u|\mathbf{1}_{A_n} > |u|\}} |u|\mathbf{1}_{A_n}\, d\mu = 0.$$

(5 pts)

(d) Show that $\lim_{n\to\infty} \int_{A_n} u\, d\mu = 0$. (5 pts)

(5) Let $E = \{(x,y) : 0 < x < \infty, 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x,y) = y \sin x\, e^{-xy}$.

(a) Show that f is $\lambda \times \lambda$ integrable on E. (8 pts)

(b) Applying Fubini's Theorem to the function f, show that

$$\int_0^\infty \frac{\sin x}{x} \left(\frac{1 - e^{-x}}{x} - e^{-x}\right) dx = \frac{1}{2} \log 2.$$

(12pts)