

Measure and Integration: Hertentamen 2013-14

- (1) Consider the measure space $([0, 1], \mathcal{B}([0, 1]), \lambda)$, where $\mathcal{B}([0, 1])$ is the restriction of the Borel σ -algebra to $[0, 1]$, and λ is the restriction of Lebesgue measure to $[0, 1]$. Let E_1, \dots, E_m be a collection of Borel measurable subsets of $[0, 1]$ such that every element $x \in [0, 1]$ belongs to at least n sets in the collection $\{E_j\}_{j=1}^m$, where $n \leq m$. Show that there exists a $j \in \{1, \dots, m\}$ such that $\lambda(E_j) \geq \frac{n}{m}$. (1.5 pt)
- (2) Let (X, \mathcal{F}, μ) be a measure space, and $1 < p, q < \infty$ conjugate numbers, i.e. $1/p + 1/q = 1$. Show that if $f \in \mathcal{L}^p(\mu)$, then there exists $g \in \mathcal{L}^q(\mu)$ such that $\|g\|_q = 1$ and $\int fg d\mu = \|f\|_p$. (1.5 pt)
- (3) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra and λ is Lebesgue measure. Let $f \in \mathcal{L}^1(\lambda)$ and define for $h > 0$, the function $f_h(x) = \frac{1}{h} \int_{[x, x+h]} f(t) d\lambda(t)$.
- (a) Show that f_h is Borel measurable for all $h > 0$. (1 pt)
- (b) Show that $f_h \in \mathcal{L}^1(\lambda)$ and $\|f_h\|_1 \leq \|f\|_1$. (1 pt)
- (4) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra, and λ Lebesgue measure.
- (a) Show that for any $u \in \mathcal{L}^1(\lambda)$, one has $u \mathbf{1}_{[n-1/n, n+1/n]} \xrightarrow{\lambda} 0$. (1 pt)
- (b) Show that for any $u \in \mathcal{L}^1(\lambda)$, the sequence $(|u| \mathbf{1}_{[n-1/n, n+1/n]})$ is uniform integrable. (1 pt)
- (c) Show that for any $u \in \mathcal{L}^1(\lambda)$ one has, $\lim_{n \rightarrow \infty} \int_{[n-1/n, n+1/n]} u d\lambda = 0$. (1 pt)
- (5) Let (X, \mathcal{A}, μ) be a measure space and $f \in \mathcal{L}^1(\mu)$. Define $A_n = \{x \in X : 1/n \leq |f(x)| < n\}$, for $n \geq 1$. Show that for every $\epsilon > 0$, there exists a positive integer N , such that $\mu(A_N) < \infty$ and $\int_{A_N^c} |f| d\mu < \epsilon$. (2 pts)

