Measure and Integration: Solutions Retake Final 2013-14

(1) Consider the measure space $([0,1]\mathcal{B}([0,1]),\lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra to $[0,1]$, and λ is the restriction of Lebesgue measure to $[0,1]$. Let $E_1,\cdots,E_m$ be a collection of Borel measurable subsets of $[0,1]$ such that every element $x\in[0,1]$ belongs to at least n sets in the collection $\{E_j\}_{j=1}^m$, where $n\le m$. Show that there exists a $j\in\{1,\cdots,m\}$ such that $\lambda(E_j)\ge \dfrac{n}{m}$. (1.5 pt)

Solution: By hypothesis, for any $x\in[0,1]$ we have $\displaystyle\sum_{j=}^{m}\mathbf{1}_{E_j}(x)\ge n$. Assume for the sake of getting a contradiction that $\lambda(E_j)<\dfrac{n}{m}$ for all $1\le j\le m$. Then,

$$n=\int_{[0,1]} n\,d\lambda\le\int\sum_{j=}^{m}\mathbf{1}_{E_j}(x)\,d\lambda=\sum_{j=1}^{m}\lambda(E_j)<\sum_{j=}^{m}\frac{n}{m}=n,$$

a contradiction. Hence, there exists $j\in\{1,\cdots,m\}$ such that $\lambda(E_j)\ge\dfrac{n}{m}$.

(2) Let $(X,\mathcal{F},\mu)$ be a measure space, and $1<p,q<\infty$ conjugate numbers, i.e. $1/p+1/q=1$. Show that if $f\in\mathcal{L}^p(\mu)$, then there exists $g\in\mathcal{L}^q(\mu)$ such that $||g||_q=1$ and $\int fg\,d\mu=||f||_p$. (1.5 pt)

Solution: Note that $q(p-1)=p$, so we define $g=\mathrm{sgn}(f)\left(\dfrac{f}{||f||_p}\right)^{p-1}$. Then,

$$\int|g|^q\,d\mu=\int\frac{|f|^p}{||f||_p^p}\,d\mu=1.$$

So $||g||_q=1$ and

$$\int fg\,d\mu=\int|fg|\,d\mu=\int\frac{|f|^p}{||f||_p^{p-1}}\,d\mu=||f||_p.$$

(3) Consider the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$, where $\mathcal{B}(\mathbb{R}))$ is the Borel σ-algebra and λ is Lebesgue measure. Let $f\in\mathcal{L}^1(\lambda)$ and define for $h>0$, the function $f_h(x)=\dfrac{1}{h}\displaystyle\int_{[x,x+h]}f(t)\,d\lambda(t)$.

(a) Show that f_h is Borel measurable for all $h>0$. (1 pt)
(b) Show that $f_h\in\mathcal{L}^1(\lambda)$ and $||f_h||_1\le||f||_1$. (1 pt)

Solution (a): For $h>0$, define $u_h(t,x)=\frac{1}{h}\mathbf{1}_{[x,x+h]}(t)f(t)$, then u_h is $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$ measurable. Applying Tonelli's Theorem (Theorem 13.8(ii)) to the positive and negative parts of the function u_h, we have that the functions

$$x\to\int u^+(t,x)\,d\lambda(t)=f_h^+(x),\ \text{and}\ x\to\int u^-(t,x)\,d\lambda(t)=f_h^-(x)$$

are $\mathcal{B}(\mathbb{R})$ measurable. Hence, f_h is Borel measurable for all $h>0$.

Solution (b): Note that

$$\int\int\frac{1}{h}\mathbf{1}_{[x,x+h]}(t)|f(t)|\,d\lambda(x)\,d\lambda(t)=\int\int\frac{1}{h}\mathbf{1}_{[t-h,t]}(x)|f(t)|\,d\lambda(x)\,d\lambda(t)=\int|f(t)|\,d\lambda(t)<\infty.$$

Hence, by Fubini's Theorem $f_h\in\mathcal{L}^1(\lambda)$ and

$$\int|f_h(x)|\,d\lambda(x)=\int\int\frac{1}{h}\mathbf{1}_{[x,x+h]}(t)|f(t)|\,d\lambda(x)\,d\lambda(t)=\int|f(x)|\,d\lambda(x)=||f||_1.$$

(4) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure.

(a) Show that for any $u \in \mathcal{L}^1(\lambda)$, one has $u\mathbf{1}_{[n-1/n,n+1/n]} \xrightarrow{\lambda} 0$. (1 pt)

(b) Show that for any $u \in \mathcal{L}^1(\lambda)$, the sequence $(|u|\mathbf{1}_{[n-1/n,n+1/n]})$ is uniform integrable. (1 pt)

(c) Show that for any $u \in \mathcal{L}^1(\lambda)$ one has, $\lim_{n\to\infty} \int_{[n-1/n,n+1/n]} u\, d\lambda = 0$. (1 pt)

Proof (a): Let $u \in \mathcal{L}^1(\lambda)$, and set $A_n = [n - 1/n, n + 1/n]$. For any $\epsilon > 0$ and any$A \in \mathcal{A}$ with $\lambda(A) < \infty$, one has

$$\lambda(A \cap \{|u|\mathbf{1}_{A_n} > \epsilon\}) = \lambda(A \cap A_n \cap \{|u| > \epsilon\}) \leq \lambda(A_n) = 1/2n.$$

This shows that $\lim_{n\to\infty} \lambda(A \cap \{|u|\mathbf{1}_{A_n} > \epsilon\}) = 0$, and hence $u\mathbf{1}_{A_n} \xrightarrow{\lambda} 0$.

Proof (b): Let $u \in \mathcal{L}^1(\lambda)$. Note that $|u|\mathbf{1}_{A_n} \leq |u|$, thus the set $\{|u|\mathbf{1}_{A_n} > |u|\}$ is empty. By Theorem 10.9(ii), we have

$$\int_{\{|u|\mathbf{1}_{A_n} > |u|\}} |u|\mathbf{1}_{A_n}\, d\lambda = 0$$

for all n and hence $\sup_n \int_{\{|u|\mathbf{1}_{A_n} > |u|\}} |u|\mathbf{1}_{A_n}\, d\lambda = 0$. Taking $w_\epsilon = |u|$ for any ϵ we see that the sequence $(|u|\mathbf{1}_{A_n})$ is uniformly integrable.

Proof (c): By part (b), the sequence $(|u|\mathbf{1}_{A_n})$ is uniformly integrable. Hence, by part (a) and Vitali's Theorem 16.6 we have

$$\lim_{n\to\infty} \int |u|\mathbf{1}_{A_n}\, d\lambda = \lim_{n\to\infty} ||u\mathbf{1}_{A_n}||_1 = 0.$$

Since

$$\limsup_{n\to\infty} |\int u\mathbf{1}_{A_n}\, d\lambda| \leq \limsup_{n\to\infty} \int |u|\mathbf{1}_{A_n}\, d\lambda = \lim_{n\to\infty} \int |u|\mathbf{1}_{A_n}\, d\lambda = 0$$

the result follows.

(5) Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in \mathcal{L}^1(\mu)$. Define $A_n = \{x \in X : 1/n \leq |f(x)| < n\}$, for $n \geq 1$. Show that for every $\epsilon > 0$, there exists a positive integer N, such that $\mu(A_N) < \infty$ and $\int_{A_N^c} |f|\, d\mu < \epsilon$. (2 pts)

Solution: Note that $A_1 \subseteq A_2 \subseteq \cdots$ and $\bigcup_{n=1}^\infty A_n = \{x \in X : 0 < |f(x)| < \infty\}$. Define

$$A_0 = \{x \in X : f(x) = 0\}, \quad \text{and } A_\infty = \{x \in X : |f(x)| = \infty\},$$

and set $A = \bigcup_{n=1}^\infty A_n$, then, $X = A_0 \cup A \cup A_\infty$ is a disjoint union. By integrability of f, we have $\mu(A_\infty) = 0$ and since $f = 0$ on A_0, we $\int |f|\, d\mu = \int_A |f|\, d\mu$. Since $|f|\mathbf{1}_{A_n} \nearrow |f|\mathbf{1}_A$, we get by Beppo-Levi that

$$\lim_{n\to\infty} \int_{A_n} |f|\, d\mu = \int_A |f|\, d\mu = \int |f|\, d\mu.$$

Now given $\epsilon > 0$, choose N sufficiently large so that

$$\int_{A_N} |f|\, d\mu \geq \int |f|\, d\mu - \epsilon = \int_{A_N} |f|\, d\mu + \int_{A_N^c} |f|\, d\mu - \epsilon.$$

Since $\int_{A_N} |f|\, d\mu < \infty$, this implies that $\int_{A_N^c} |f|\, d\mu < \epsilon$. By definition of A_N we get

$$\mu(A_N) = \int \mathbf{1}_{A_N}\, d\mu \leq \int_{A_N} N|f|\, d\mu = N \int |f|\, d\mu < \infty.$$