Measure and Integration: Solution Final 2015-16

(1) Consider the measure space $[0,1],\mathcal{B}([0,1]),\lambda)$ where λ is Lebesgue measure on $[0,1]$. Define $u_n(x)=\dfrac{n^2x^2}{1+n^2x}$ for $x\in[0,1]$ and $n\geq 1$. Show that

$$\lim_{n\to\infty}\int_{[0,1]}\frac{n^2x^2}{1+n^2x}\,d\lambda(x)=1/2.$$

(1 pt)

Proof: Note that u_n is continuous, hence measurable. Furtheremore $\lim\limits_{n\to\infty}u_n(x)=x$ for all $x\in[0,1]$. For $x\in[0,1]$, one has $n^2x^2<1+n^2x$, so that $u_n(x)\leq 1$, and $1\in\mathcal{L}^1_+(\lambda)$. By Lebesgue Dominated Convergence Theorem, and the fact that the function $f(x)=x$ is Riemann-integrable,

$$\lim_{n\to\infty}\int_{[0,1]}\frac{n^2x^2}{1+n^2x}\,d\lambda(x)=\int_{[0,1]}\lim_{n\to\infty}\frac{n^2x^2}{1+n^2x}\,d\lambda(x)=\int_{[0,1]}x\,d\lambda=1/2.$$

(1.5 pts)

(2) Suppose μ and ν are finite measures on $(X,\mathcal{A})$. Show that there exists a function $f\in\mathcal{L}^1_+(\mu)$, and a set $A_0\in\mathcal{A}$ with $\mu(A_0)=0$ such that

$$\nu(E)=\int_E f\,d\mu+\nu(A_0\cap E),$$

for all $E\in\mathcal{A}$. (1.5 pts)

Proof: By Lebesgue Decomposition Theorem, there exist measures ρ and σ on $\mathcal{A}$ such that ρ is absolutely continuous with respect to μ, σ is mutually singular with respect to μ, and $\nu=\rho+\sigma$. By Radon-Nikodym Theorem, there exists $f\in\mathcal{L}^1_+(\mu)$ such that $\rho(B)=\int_B f\,d\mu$. By mutual singularity of σ and μ, there exists a set $A_0\in\mathcal{A}$ such that $\mu(A_0)=\sigma(A_0^c)=0$. For any $E\in\mathcal{A}$,

$$\nu(E)=\rho(E)+\sigma(E)=\int_E f\,d\mu+\sigma(E).$$

Since $\sigma(A_0^c\cap E)=0$, we have $\sigma(E)=\sigma(A_0\cap E)$. Also since $\mu(A_0\cap E)=0$, then by absolute continuity of ρ we have $\rho(A_0\cap E)=0$, hence $\nu(A_0\cap E)=\sigma(A_0\cap E)=\sigma(E)$. Therefore,

$$\nu(E)=\rho(E)+\sigma(E)=\int_E f\,d\mu+\nu(A_0\cap E).$$

(3) Consider the measure space $[0,1),\mathcal{B}([0,1)),\lambda)$ where λ is Lebesgue measure on $[0,1)$. Let $D_1=[0,1/2)$ and $D_k=\left[\sum_{i=1}^{k-1}2^{-i},\sum_{i=1}^{k}2^{-i}\right)$, $k\geq 2$. Define $u(x)=\sqrt{2^{k-1}}$ for $x\in D_k$, $k\geq 1$. Determine the values of $p\in[1,\infty)$ such that $u\in\mathcal{L}^p(\lambda)$. In case $u\in\mathcal{L}^p(\lambda)$, find the value of $||u||_p$. (2 pts.)

Proof Note that $u=\sum_{k=1}^{\infty}\sqrt{2^{k-1}}\mathbf{1}_{D_k}>0$. By Corollary 9.9, we have

$$\int|u|^p\,d\lambda=\int\sum_{k=1}^{\infty}2^{(k-1)p/2}\mathbf{1}_{D_k}\,d\lambda=\sum_{k=1}^{\infty}2^{(k-1)p/2}2^{-k}=2^{-p/2}\sum_{k=1}^{\infty}\left(2^{(p/2-1)}\right)^k.$$

The latter series is geometric and hence convergent if $2^{(p/2-1)}<1$, equivalently $p<2$. Since $p\geq 1$ we have $u\in\mathcal{L}^p(\lambda)$ for $1\leq p<2$. In this case we have

$$||u||_p^p=\int|u|^p\,d\lambda=2^{-p/2}2^{p/2-1}\frac{1}{1-2^{(p/2-1)}}=\frac{1}{2-2^{p/2}}.$$

Hence,

$$||u||_p = \left(\frac{1}{2-2^{p/2}}\right)^{1/p}.$$

(4) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and Let $(u_j)_j \subseteq \mathcal{L}^1(\mu)$. Suppose $(u_j)_j$ converges to u μ a.e., and that the sequence (u_j^-) is uniformly integrable. Prove that $\liminf_{n\to\infty} \int u_n \, d\mu \geq \int u \, d\mu$. (2 pts)

Proof Since (u_j) converges to u μ a.e., then $(u_j^-)_j$ converges μ a.e. and hence in μ measure to u^-. By Vitali's Theorem applied to the sequence $(u_j^-)_j$, we have that $(u_j^-)_j$ converges to u^- in $\mathcal{L}^1(\mu)$, and hence $\lim_{j\to\infty} \int u_j^- \, d\mu = \int u^- \, d\mu$, and $u \in \mathcal{L}^1(\mu)$. From the above we have

$$\liminf_{j\to\infty} \int u_j \, d\mu = \liminf_{j\to\infty} \int u_j^+ \, d\mu - \lim_{j\to\infty} \int u_j^- \, d\mu = \liminf_{j\to\infty} \int u_j^+ \, d\mu - \int u^- \, d\mu.$$

Since (u_j^+) converges to u^+ μ a.e., by Fatous Lemma

$$\liminf_{j\to\infty} \int u_j^+ \, d\mu \geq \int \liminf_{j\to\infty} u_j^+ \, d\mu = \int u^+ \, d\mu.$$

From the above we have

$$\liminf_{j\to\infty} \int u_j \, d\mu \geq \int u^+ \, d\mu - \int u^- \, d\mu = \int u \, d\mu.$$

(5) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and assume $u \in \mathcal{M}^+(\mathcal{A})$. Let $\phi : [0,\infty) \to \mathbb{R}$ be continuously differentiable (i.e. ϕ' exists and is continuous) such that $\phi(0) = 0$ and $\phi' \geq 0$ for all $t \geq 0$. Show that

$$\int_X \phi \circ u(x) \, d\mu = \int_{[0,\infty)} \phi'(t)\mu(\{x \in X : u(x) \geq t\}) \, d\lambda(t).$$

Conclude that if $u \in \mathcal{L}^p_+(\mu)$, then

$$\int_X u^p \, d\mu = p \int_{[0,\infty)} t^{p-1}\mu(\{x \in X : u(x) \geq t\}) \, d\lambda(t).$$

(2 pts)

Proof Note that for each $r \geq 0$, the function $\phi'(t)$ is Riemann-integrable on $[0, r]$, and by the Fundamental Theorem of Calculus we have

$$\int_{[0,r]} \phi'(t) \, d\lambda(t) = (R) \int_0^r \phi'(t)dt = \phi(r).$$

The set $E = \{(x,t) \in X \times [0,\infty) : u(x) \geq t\}$ is measurable, and the function $\phi'(t)$ is continuous and thus measurable. Since the product of two measurable functions is measurable, the function $\phi'(t)\mathbf{1}_{E(x,t)}$ is measurable. By Tonelli's Theorem, and the above we have

$$\begin{aligned}
\int_{[0,\infty)} \phi'(t)\mu(\{x \in X : u(x) \geq t\}) \, d\lambda(t) &= \int_{[0,\infty)} \int_X \phi'(t)\mathbf{1}_{\{x:t\leq u(x)\}}(x) \, d\mu(x) \, d\lambda(t) \\
&= \int_{[0,\infty)} \int_X \phi'(t)\mathbf{1}_{E(x,t)}(x,t) \, d\mu(x) \, d\lambda(t) \\
&= \int_X \int_{[0,\infty)} \phi'(t)\mathbf{1}_{\{t:t\leq u(x)\}}(t) \, d\lambda(t) \, d\mu(x) \\
&= \int_X \int_{[0,u(x)]} \phi'(t) \, d\lambda(t) \, d\mu(x) \\
&= \int_X \phi(u(x)) \, d\mu(x) = \int_X \phi \circ u(x) \, d\mu(x).
\end{aligned}$$

Finally, let $\phi(t) = t^p$ for $t \geq 0$, then ϕ is continuously differnetiable and $\phi'(t) = pt^{p-1}$. Thus.

$$\int_X u^p \, d\mu = \int_X \phi \circ u \, d\mu = p \int_{[0,\infty)} t^{p-1} \mu(\{x \in X : u(x) \geq t\}) \, d\lambda(t).$$

(6) Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in \mathcal{L}^1(\mu) \cap \mathcal{L}^2(\mu)$.
  (a) Show that $f \in \mathcal{L}^p(\mu)$ for all $1 \leq p \leq 2$. (1 pt)
  (b) Prove that $\lim\limits_{p \searrow 1} ||f||_p^p = ||f||_1$. (1 pt)

Proof (a): Let $A = \{x \in X : |f(x)| \geq 1\}$, then $A \in \mathcal{A}$ and hence the function $g = |f|^2 \mathbf{1}_A + |f| \mathbf{1}_{A^c}$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable. Furthermore

$$\int |g| \, d\mu = \int_A |f|^2 \, d\mu + \int_{A^c} |f| \, d\mu \leq ||f||_2 + ||f||_1 < \infty.$$

Thus $g \in \mathcal{L}^1(\mu)$. Now, let $1 \leq p \leq 2$, then $|f(x)|^p \leq |f(x)|^2$ for $x \in A$. and $|f(x)|^p \leq |f(x)|$ for $x \in A^c$. Thus $|f(x)|^p \leq g(x)$ for all $x \in X$ implying that $|f|^p \in \mathcal{L}^1(\mu)$, equivalently $f \in \mathcal{L}^p(\mu)$.

Proof (b): Let $(p_n)_n$ be a sequence in $[1, 2]$ with $\lim\limits_{n \to \infty} p_n = 1$. Note that for each $x \in X$, we have $\lim\limits_{n \to \infty} |f(x)|^{p_n} = |f(x)|$, and $|f(x)|^{p_n} \leq g(x)$, where g is the function defined in part (a). Since $g \in \mathcal{L}^1(\mu)$, then by Lebesgue Dominated Convergence Theorem we have,

$$\lim_{n \to \infty} ||f||_{p_n}^{p_n} = \lim_{n \to \infty} \int |f|^{p_n} \, d\mu = \int \lim_{n \to \infty} |f|^{p_n} \, d\mu = \int |f| \, d\mu = ||f||_1.$$

Thus, $\lim\limits_{p \searrow 1} ||f||_p^p = ||f||_1$.