Measure and Integration: Quiz 2015-16

1. Consider the measure space $([0,1), \mathcal{B}([0,1)), \lambda)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra restricted to $[0,1)$ and λ is the restriction of Lebesgue measure on $[0,1)$. Define the transformation $T : [0,1) \to [0,1)$ given by

$$T(x) = \begin{cases} 3x & 0 \le x < 1/3, \\ \frac{3}{2}x - \frac{1}{2}, & 1/3 \le x < 1. \end{cases}$$

   (a) Show that T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable, and determine the image measure $T(\lambda) = \lambda \circ T^{-1}$. (1 pt.)

   (b) Let $\mathcal{C} = \{A \in \mathcal{B}([0,1)) : \lambda(T^{-1}A \Delta A) = 0\}$. Show that $\mathcal{C}$ is a σ-algebra. (Note that $T^{-1}A\Delta A = \Big(T^{-1}A \setminus A\Big) \cup \Big(A \setminus T^{-1}A\Big)$). (1 pt.)

   (c) Suppose $A \in \mathcal{B}([0,1))$ satisfies the property that $T^{-1}(A) = A$ and $0 < \lambda(A) < 1$. Define μ_1, μ_2 on $\mathcal{B}([0,1))$ by

$$\mu_1(B) = \frac{\lambda(A \cap B)}{\lambda(A)}, \text{ and } \mu_2(B) = \frac{\lambda(A^c \cap B)}{\lambda(A^c)}.$$

   Show that μ_1, μ_2 are measures on $\mathcal{B}([0,1))$ satisfying

   (i) $T(\mu_i) = \mu_i, \ i = 1, 2,$

   (ii) $\lambda = \alpha\mu_1 + (1-\alpha)\mu_2$ for an appropriate $0 < \alpha < 1$.

   (1.5 pts.)

2. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$, and λ is Lebesgue measure. Define f on $\mathbb{R}$ by $f(x) = 2x\mathbf{1}_{[0,1)}(x)$.

   (a) Show that f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. (1 pt.)

   (b) Find a sequence (f_n) in $\mathcal{E}^+$ such that $f_n \nearrow f$. (1 pt.)

   (c) Determine the value of $\int f \, d\mu$ using only the material of Chapter 9. (1 pt.)

   (d) Let $\mathcal{C} = \sigma(\{\{x\} : x \in [0,1)\})$ and $\mathcal{A} = \{A \subseteq [0,2) : A \text{ is countable or } A^c \text{ is countable}\}$. Show that f is $\mathcal{C}/\mathcal{A}$ measurable and $\mathcal{C} = \mathcal{A} \cap [0,1)$. (Here we are seeing f as a function defined on $[0,1)$) (1 pt.)

3. Consider the measure space $([0,1]\mathcal{B}([0,1]), \lambda)$, where λ is the restriction of Lebesgue measure to $[0,1]$, and let $A \in \mathcal{B}([0,1])$ be such that $\lambda(A) = 1/2$. Consider the real function f defined on $[0,1]$ by $f(x) = \lambda\Big(A \cap [0,x]\Big)$.

   (a) Show that for any $x, y \in [0,1]$, we have

$$|f(x) - f(y)| \le |x - y|.$$

   Conclude that f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. (1 pt.)

   (b) Show that for any $\alpha \in (0, 1/2)$, there exists $A_\alpha \subset A$ with $A_\alpha \in \mathcal{B}([0,1])$ and $\lambda(A_\alpha) = \alpha$. (1.5 pts.)