Measure and Integration: Solution Quiz 2015-16

1. Consider the measure space $([0,1), \mathcal{B}([0,1)), \lambda)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra restricted to $[0,1)$ and λ is the restriction of Lebesgue measure on $[0,1)$. Define the transformation $T : [0,1) \to [0,1)$ given by

$$T(x) = \begin{cases} 3x & 0 \le x < 1/3, \\ \frac{3}{2}x - \frac{1}{2}, & 1/3 \le x < 1. \end{cases}$$

(a) Show that T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable, and determine the image measure $T(\lambda) = \lambda \circ T^{-1}$. (1 pt.)

(b) Let $\mathcal{C} = \{A \in \mathcal{B}([0,1)) : \lambda(T^{-1}A\Delta A) = 0\}$. Show that $\mathcal{C}$ is a σ-algebra. (Note that $T^{-1}A\Delta A = \Big(T^{-1}A \setminus A\Big) \cup \Big(A \setminus T^{-1}A\Big)$). (1 pt.)

(c) Suppose $A \in \mathcal{B}([0,1))$ satisfies the property that $T^{-1}(A) = A$ and $0 < \lambda(A) < 1$. Define μ_1, μ_2 on $\mathcal{B}([0,1))$ by

$$\mu_1(B) = \frac{\lambda(A \cap B)}{\lambda(A)}, \text{ and } \mu_2(B) = \frac{\lambda(A^c \cap B)}{\lambda(A^c)}.$$

Show that μ_1, μ_2 are measures on $\mathcal{B}([0,1))$ satisfying

(i) $T(\mu_i) = \mu_i,\ i = 1,2,$

(ii) $\lambda = \alpha\mu_1 + (1-\alpha)\mu_2$ for an appropriate $0 < \alpha < 1$.

(1 pt.)

Solution(a): To show T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable, it is enough to consider inverse images of intervals of the form $[a,b) \subset [0,1)$. Now,

$$T^{-1}([a,b)) = [\frac{a}{3}, \frac{b}{3}) \cup [\frac{2a+1}{3}, \frac{2b+1}{3}) \in \mathcal{B}([0,1)).$$

Thus, T is measurable.

We claim that $T(\lambda) = \lambda$. To prove this, we use Theorem 5.7. Notice that $\mathcal{B}([0,1))$ is generated by the collection $\mathcal{G} = \{[a,b) : 0 \le a \le b < 1\}$ which is closed under finite intersections. Now,

$$\begin{aligned} T(\lambda)([a,b)) &= \lambda(T^{-1}([a,b))) \\ &= \lambda([\frac{a}{3}, \frac{b}{3})) + \lambda([\frac{2a+1}{3}, \frac{2b+1}{3})) \\ &= b - a = \lambda([a,b)). \end{aligned}$$

Since the constant sequence $([0,1))$ is exhausting, belongs to $\mathcal{G}$ and $\lambda([0,1)) = T(\lambda([0,1)) = 1 < \infty$, we have by Theorem 5.7 that $T(\lambda) = \lambda$.

Solution(b): We check the three conditions for a collection of sets to be a σ-algebra. Firstly, the empty set $\emptyset \in \mathcal{B}([0,1))$ and $T^{-1}(\emptyset) = \emptyset$, hence $\lambda(T^{-1}\emptyset\Delta\emptyset) = \lambda(\emptyset) = 0$, so $\emptyset \in \mathcal{C}$. Secondly, Let $A \in \mathcal{C}$, then $\lambda(T^{-1}A\Delta A) = 0$. Since

$$\lambda(T^{-1}A^c\Delta A^c) = \lambda(T^{-1}A\Delta A) = 0,$$

and $A^c \in \mathcal{B}([0,1))$, we have $A^c \in \mathcal{C}$. Thirdly, let (A_n) be a sequence in $\mathcal{C}$, then $A_n \in \mathcal{B}([0,1))$ and $\lambda(T^{-1}A_n\Delta A_n) = 0$ for each n. Since $\mathcal{B}([0,1))$ is a σ-algebra, we have $\bigcup_n A_n \in \mathcal{B}([0,1))$, and

$$T^{-1}(\bigcup_n A_n) = \bigcup_n T^{-1}A_n = \bigcup_n A_n.$$

An easy calculation shows that

$$T^{-1}(\bigcup_n A_n)\Delta\bigcup_m A_m \subseteq \bigcup_n (T^{-1}A_n\Delta A_n).$$

By σ-subadditivity of λ, we have

$$\lambda\Big(T^{-1}(\bigcup_m A_m)\Delta\bigcup_n A_n\Big) \leq \sum_n \lambda\Big(T^{-1}A_n\Delta A_n\Big) = 0.$$

Thus, $\bigcup_n A_n \in \mathcal{C}$. This shows that $\mathcal{C}$ is a σ-algebra.

Solution (c): First note that $0 < \lambda(A^c) < 1$ and $T^{-1}(A^c) = A^c$. The proofs that μ_1 and μ_2 are measures are similar, so we only prove that μ_1 is a measure. First note that

$$\mu_1(\emptyset) = \frac{\lambda(A\cap\emptyset)}{\lambda(A)} = \frac{\lambda(\emptyset)}{\lambda(A)} = 0.$$

Suppose (A_i) is a pairwise disjoint sequence in $\mathcal{B}([0,1))$. Then

$$\mu_1(\bigcup_i A_i) = \frac{\lambda(A\cap\bigcup_i A_i)}{\lambda(A)} = \sum_i \frac{\lambda(A\cap A_i)}{\lambda(A)} = \sum_i \mu_1(A_i).$$

Hence, μ_1 is a measure. A similar proof shows that μ_2 is a measure. We now show (i). Firstly, since $T^{-1}(A) = A$ and $T^{-1}(A^c) = A^c$ we have by (a),

$$\lambda(A\cap T^{-1}(B)) = \lambda(T^{-1}(A)\cap T^{-1}(B)) = \lambda(T^{-1}(A\cap B) = \lambda(A\cap B),$$

and

$$\lambda(A^c\cap T^{-1}(B)) = \lambda(T^{-1}(A^c)\cap T^{-1}(B)) = \lambda(T^{-1}(A^c\cap B) = \lambda(A^c\cap B),$$

for any $B \in \mathcal{B}([0,1))$. Thus,

$$T(\mu_1)(B) = \mu_1(T^{-1}(B)) = \frac{\lambda(A\cap T^{-1}(B))}{\lambda(A)} = \frac{\lambda(A\cap B)}{\lambda(A)} = \mu_1(B),$$

and

$$T(\mu_2)(B) = \mu_2(T^{-1}(B)) = \frac{\lambda(A^c\cap T^{-1}(B))}{\lambda(A^c)} = \frac{\lambda(A^c\cap B)}{\lambda(A^c)} = \mu_2(B).$$

To prove (ii), we notice that for any $B \in \mathcal{B}([0,1))$,

$$\lambda(B) = \lambda(A\cap B) + \lambda(A^c\cap B) = \lambda(A)\mu_1(B) + \lambda(A^c)\mu_2(B).$$

Since $\lambda(A^c) = 1 - \lambda(A)$, the result follows with $\alpha = \lambda(A)$.

2. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$, and λ is Lebesgue measure. Define f on $\mathbb{R}$ by $f(x) = 2x\mathbf{1}_{[0,1)}(x)$.

   (a) Show that f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. (1 pt.)

   (b) Find a sequence (f_n) in $\mathcal{E}^+$ such that $f_n \nearrow f$. (1 pt.)

   (c) Determine the value of $\int f \, d\mu$ using only the material of Chapter 9. (1 pt.)

   (d) Let $\mathcal{C} = \sigma(\{\{x\} : x \in [0,1)\})$ and $\mathcal{A} = \{A \subseteq [0,2) : A \text{ is countable or } A^c \text{ is countable}\}$. Show that f is $\mathcal{C}/\mathcal{A}$ measurable and $\mathcal{C} = \mathcal{A} \cap [0,1)$. (Here we are seeing f as a function defined on $[0,1)$) (1 pt.)

Solution(a): Note that the function $g(x) = 2x$ is continuous and hence $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. Also $[0,1) \in \mathcal{B}(\mathbb{R})$, hence $\mathbf{1}_{[0,1)}$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. Finally f is the product of two measurable functions, and therefore f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable.

Solution(b): Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \sum_{k=0}^{2^n-1} \frac{2k}{2^n} \cdot 1_{[k/2^n,(k+1)/2^n)}, \; n \geq 1.$$

Since $[k/2^n, (k+1)/2^n) \in \mathcal{B}(\mathbb{R})$, then $f_n \in \mathcal{E}^+$. We now show that f_n increases to f. For $x \notin [0,1)$, i.e have $f_n(x) = f_{n+1}(x) = 0$. Suppose $x \in [0,1)$, then there exists a $0 \leq k \leq 2^n - 1$ such that $x \in [k/2^n, (k+1)/2^n)$. Since

$$[k/2^n, (k+1)/2^n) = [2k/2^{n+1}, (2k+1)/2^{n+1}) \cup [(2k+1)/2^{n+1}, (2k+2)/2^{n+1}),$$

we see that $f_n(x) = \frac{2k}{2^n}$ while $f_{n+1}(x) \in \{\frac{4k}{2^{n+1}}, \frac{2(2k+1)}{2^{n+1}}\}$ so that $f_n(x) \leq f_{n+1}(x)$.
For $x \notin [0,1)$, we have $f(x) = f_n(x) = 0$ for all n. For $x \in [0,1)$, there exists for each n, an integer $k_n \in \{0, 1, \cdots, 2^n - 1\}$ such that $x \in [k_n/2^n, (k_n+1)/2^n)$. Thus,

$$|f(x) - f_n(x)| = |2x - \frac{2k_n}{2^n}| = 2|x - \frac{k_n}{2^n}| < \frac{1}{2^{n-1}}.$$

Since f_n is an increasing sequence, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \sup_n f_n(x).$$

Solution(c): Since f is the supremum of measurable functions, by Corollary 8.9 f is measurable. To calculate the integral we apply Beppo-Levi,

$$\begin{aligned}
\int f \, d\lambda &= \lim_{n\to\infty} \int f_n \, d\lambda \\
&= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \frac{2k}{2^n} \lambda([k/2^n, (k+1)/2^n)) \\
&= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \frac{2k}{2^n} \frac{1}{2^n} \\
&= \lim_{n\to\infty} \frac{2}{4^n} \sum_{k=0}^{2^n-1} k \\
&= \lim_{n\to\infty} \frac{2}{2} \frac{(2^n-1)2^n}{4^n} = 1.
\end{aligned}$$

Solution(d): First note that $\mathcal{A}$ is a σ-algebra. Let $A \in \mathcal{A}$, and set $B = \{a/2 : a \in A\}$. Then $f^{-1}(A) = B$. If A is countable, then B is countable and can be written as a countable union of the form $B = \cup_{x\in B}\{x\}$. Since $\mathcal{C}$ is a σ-algebra and $\{x\} \in \mathcal{C}$ we have $A \in \mathcal{C}$, and $f^{-1}(A) = B \in \mathcal{C}$. Similarly if A^c is countable, then $B^c = \{a/2 : a \in A^c\}$ is countable and can be written as a countable union of the form $B^c = \cup_{x\in B^c}\{x\}$, hence $A^c \in \mathcal{C}$, and $B^c \in \mathcal{C}$. Since $B^c = f^{-1}(A^c) = (f^{-1}(A))^c$, we see that $B = f^{-1}(A) \in \mathcal{C}$. Thus, f is $\mathcal{C}/\mathcal{A}$ measurable. Now a similar argument as above shows that if $A \in \mathcal{A} \cap [0,1))$ is countable then $A \in \mathcal{C}$, and if A^c is countable then $A^c \in \mathcal{C}$. Thus $\mathcal{A} \cap [0,1) \subseteq \mathcal{C}$. Finally, notice that the generators of $\mathcal{C}$ are elements of $\mathcal{A} \cap [0,1)$, hence $\mathcal{C} \subseteq \mathcal{A} \cap [0,1)$ implying equality.

3. Consider the measure space $([0,1]\mathcal{B}([0,1]), \lambda)$, where λ is the restriction of Lebesgue measure to $[0,1]$, and let $A \in \mathcal{B}([0,1])$ be such that $\lambda(A) = 1/2$. Consider the real function f defined on $[0,1]$ by $f(x) = \lambda\Big(A \cap [0,x]\Big)$.

   (a) Show that for any $x, y \in [0,1]$, we have
   $$|f(x) - f(y)| \leq |x - y|.$$
   Conclude that f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. (1 pt.)

   (b) Show that for any $\alpha \in (0,1/2)$, there exists $A_\alpha \subset A$ with $A_\alpha \in \mathcal{B}([0,1))$ and $\lambda(A_\alpha) = \alpha$. (1 pt.)

Solution(a): Let $x, y \in [0,1]$ and assume with no loss of generality that $y < x$. Then
$$|f(x) - f(y)| = f(x) - f(y) = \lambda\Big(A \cap [y,x]\Big) \leq \lambda([y,x]) = x - y = |x - y|.$$
The above shows that f is a continuous function on $[0,1]$, and hence Borel measurable.

Solution(b): Part (a) shows that f is a continuous function on $[0,1]$ with $f(0) = 0$, and $f(1) = \lambda(A) = 1/2$. Hence by the Intermediate Value Theorem, for any $\alpha \in (0,1)$ there exists $x_0 \in (0,1)$ such that $f(x_0) = \alpha$. Set $A_\alpha = A \cap [0,x_0]$, then $A_\alpha \subset A$ and $A_\alpha \in \mathcal{B}([0,1]$ satisfies $\lambda(A_\alpha) = f(x_0) = \alpha$.