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Measure and Integration: Retake Final 2015-16

(1) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and $f \in \mathcal{M}(\mathcal{A})$. Show that for every $\epsilon > 0$, there exists a set $A \in \mathcal{A}$ and $k \geq 1$ such that $\mu(A) < \epsilon$ and $|f(x)| \leq k$ for all $x \in A^c$. (1 pt)

(2) Consider the measure space $[0,1], \mathcal{B}([0,1]), \lambda)$ where λ is Lebesgue measure on $[0,1]$. Define $u_n(x) = \dfrac{nx}{1+n^2x^2}$ for $x \in [0,1]$ and $n \geq 1$. Show that

$$\lim_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2}\, d\lambda(x) = 0.$$

(1.5 pts)

(3) Let μ and ν be finite measures on $(X, \mathcal{A})$. Show that μ and ν are mutually singular **if and only if** for every $\epsilon > 0$, there exists a set $E \in \mathcal{A}$ such that $\mu(E) < \epsilon$ and $\nu(E^c) < \epsilon$. (2 pts)

(4) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $(u_n)_n \subset \mathcal{L}^p(\mu)$ converging in $\mathcal{L}^p(\mu)$ to a function $u \in \mathcal{L}^p(\mu)$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{A}$ with $\mu(A) < \delta$, then $\int_A |u_n|^p\, d\mu < \epsilon$ for all $n \geq 1$. (2 pts)

(5) Consider the measure space $([0,\infty), \mathcal{B}([0,\infty)), \lambda)$, where $\mathcal{B}([0,\infty))$ is the Borel σ-algebra, and λ is Lebesgue measure on $[0,\infty)$. Let $f(x,y) = ye^{-(1+x^2)y^2}$ for $0 \leq x, y < \infty$.

(a) Show that $f \in \mathcal{L}^1(\lambda \times \lambda)$, and determine the value of $\int_{[0,\infty)\times[0,\infty)} f\, d(\lambda\times\lambda)$. (1 pt)

(b) Prove that $\int_{[0,\infty)\times[0,\infty)} f\, d(\lambda\times\lambda) = \left(\int_{[0,\infty)} e^{-x^2}\, d\lambda(x)\right)^2$. Use part (a) to deduce the value of $\int_{[0,\infty)} e^{-x^2}\, d\lambda(x)$. (1 pt)

(6) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and Let $(u_j)_j \subseteq \mathcal{L}^p(\mu)$, $p \geq 1$. Suppose $(u_j)_j$ converges to u μ a.e., and that the sequence $((u_j^p)^-)$ is uniformly integrable. Prove that

$$\liminf_{n\to\infty} \int u_n^p\, d\mu \geq \int u^p\, d\mu.$$

(1.5 pts)