Measure and Integration: Solutions Final 2014-15

(1) Consider a measure space $(X, \mathcal{A}, \mu)$, and let $(f_n)_n$ be a sequence in $\mathcal{L}^2(\mu)$ which is bounded in the $\mathcal{L}^2$ norm, i.e. there exists a constant $C > 0$ such that $||f_n||_2 < C$ for all $n \geq 1$.

(a) Prove that $\sum_{n=1}^{\infty}(\frac{f_n}{n})^2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$. (1 pt.)

(b) Prove that $\lim_{n\to\infty} \frac{f_n}{n} = 0$ μ a.e. (1 pt.)

Proof (a): First observe that

$$\sum_{n=1}^{\infty} ||\frac{f_n}{n}||_2^2 = \sum_{n=1}^{\infty} \frac{||f_n||_2^2}{n^2} \leq \sum_{n=1}^{\infty} \frac{C^2}{n^2} < \infty.$$

Now, by Beppo-Levi and the above, we have

$$\int \sum_{n=1}^{\infty} (\frac{f_n}{n})^2 \, d\mu = \sum_{n=1}^{\infty} \int (\frac{f_n}{n})^2 \, d\mu = \sum_{n=1}^{\infty} ||\frac{f_n}{n}||_2^2 < \infty.$$

Hence, $\sum_{n=1}^{\infty}(\frac{f_n}{n})^2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$.

Proof (b): Since $\sum_{n=1}^{\infty}(\frac{f_n}{n})^2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$, then $\sum_{n=1}^{\infty}(\frac{f_n}{n})^2 < \infty$ μ a.e. and as a result $\lim_{n\to\infty}(\frac{f_n}{n}) = 0$ μ a.e.

(2) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Suppose that the real valued functions $f_n, g_n, f, g \in \mathcal{M}(\mathcal{A})$ $(n \geq 1)$ satisfy the following:

(i) $f_n \xrightarrow{\mu} f$,

(ii) $g_n \xrightarrow{\mu} g$,

(iii) $|f_n| \leq C$ for all n, where $C > 0$.

Prove that $f_n g_n \xrightarrow{\mu} fg$. (2 pts)

Proof: Let $\epsilon > 0$ and $\delta > 0$, since μ is a finite measure, it is enough to show that there exists $N \geq 1$ such that

$$\mu(\{x \in X : |f_n g_n - fg| > \epsilon\}) < \delta, \quad \text{for all } n \geq N.$$

First note that

$$|f_n g_n - fg| \leq |f_n||g_n - g| + |g||f_n - f|,$$

thus,

$$\mu(\{x \in X : |f_n g_n - fg| > \epsilon\}) \leq \mu(\{x \in X : |f_n||g_n - g| > \epsilon/2\}) + \mu(\{x \in X : |g_n||f_n - f| > \epsilon/2\}).$$

Let $E_n = \{x \in X : |g| > n\}$, then $E_1 \supseteq E_2 \supseteq \cdots$, and since g is real valued we have $\bigcap_{n=1}^{\infty} E_n = \emptyset$. By finiteness of μ, we have

$$\lim_{n\to\infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n) = 0.$$

Choose m large enough so that $\mu(E_m) < \delta/3$. By properties (i) and (ii), there exists $N \geq 1$ so that for $n \geq N$,

$$\mu(\{x \in X : |f_n - f| > \epsilon/2m\}) < \delta/3, \text{ and } \mu(\{x \in X : |g_n - g| > \epsilon/2C\}) < \delta/3.$$

Then for all $n \geq N$,

$$\mu(\{x \in X : |f_n||g_n - g| > \epsilon/2\}) \leq \mu(\{x \in X : |g_n - g| > \epsilon/2C\}) < \delta/3,$$

and

$$\mu(\{x \in X : |g||f_n - f| > \epsilon/2\}) \leq \mu(E_m) + \mu(\{x \in E_m^c : |f_n - f| > \epsilon/2m\}) < 2\delta/3.$$

Therefore, $\mu(\{x \in X : |f_n g_n - fg| > \epsilon\}) < \delta$ for all $n \geq N$, and hence $f_n g_n \xrightarrow{\mu} fg$.

(3) Let $(X, \mathcal{A})$ be a measurable space and let μ, ν be finite measures on $\mathcal{A}$.

(a) Show that there exists a function $f \in \mathcal{L}^1_+(\mu) \cap \mathcal{L}^1_+(\nu)$ such that for every $A \in \mathcal{A}$, we have

$$\int_A (1-f)\, d\mu = \int_A f\, d\nu.$$

(1 pt)

(b) Show that the function f of part (a) satisfies $0 \leq f \leq 1$ μ a.e. (1 pt)

Proof(a): First note that $\mu + \nu$ is a measure (Exercise 4.6(ii)), and that $\mu \ll \mu + \nu$. By using a standard argument (first checking indictor functions, then simple functions, then positive functions, then general integrable functions) one sees that for any $g \in \mathcal{L}^1(\mu+\nu)$ one has $g \in \mathcal{L}^1(\mu) \cap \mathcal{L}^1(\nu)$, and

$$\int g\, d(\mu+\nu) = \int g\, d\mu + \int g\, d\nu.$$

Now the condition $\int_A (1-f)\, d\mu = \int_A f\, d\nu$ is equivalent to $\mu(A) = \int_A f\, d(\mu+\nu)$. Since $\mu \ll \mu+\nu$, then by Radon-Nikodym Theorem there exists $f \in \mathcal{L}^1_+(\mu+\nu)$ such that $\mu(A) = \int_A f\, d(\mu+\nu)$. Thus, $f \in \mathcal{L}^1_+(\mu) \cap \mathcal{L}^1_+(\nu)$ and $\int_A (1-f)\, d\mu = \int_A f\, d\nu$ for all $A \in \mathcal{A}$.

Proof(b): Define ρ on $\mathcal{A}$ by $\rho(A) = \int_A f\, d\mu$ $(A \in \mathcal{A})$. Since $f \in \mathcal{L}^1_+(\nu)$, then ρ is a finite measure and $\rho \ll \nu$. By part (a), we have $\rho(A) = \int_A (1-f)\, d\mu$, $A \in \mathcal{A}$ and $(1-f) \in \mathcal{L}^1(\mu)$. By Theorem 10.9(ii), we see that if $\mu(A) = 0$, then $\rho(A) = 0$, hence $\rho \ll \mu$. By the Theorem of Radon Nikodym, there exists a unique μ a.e. function $g \in \mathcal{L}^1_+(\mu)$ such that $\rho(A) = \int_A g\, d\mu$ for all $A \in \mathcal{A}$. This gives that

$$\int_A g\, d\mu = \int_A (1-f)\, d\mu, \text{ for all } A \in \mathcal{A}.$$

By Corollary 10.14(i), we have $g = 1 - f$ μ a.e. Since $g, f \geq 0$, we get $0 \leq f \leq 1$ μ a.e.

(4) Let $0 < a < b$. Prove with the help of Tonelli's theorem (applied to the function $f(x,t) = e^{-xt}$) that $\int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t) = \log(b/a)$, where λ denotes Lebesgue measure. (2 pts)

Proof Let $f : [a,b] \times [0,\infty)$ be given by $f(x,t) = e^{-xt}$. Then f is continuous (hence measurable) and $f > 0$. By Toneli's theorem

$$\int_{[0,\infty)} \int_{[a,b]} e^{-xt} d\lambda(x)\, d\lambda(t) = \int_{[a,b]} \int_{[0,\infty)} e^{-xt} d\lambda(t)\, d\lambda(x).$$

For each fixed $x \in [a,b]$, the function $t \to e^{-xt}$ is positive measurable and the improper Riemann integrable on $[0,\infty)$ exists, so that

$$\int_{[0,\infty)} e^{-xt} d\lambda(t) = \int_0^\infty e^{-xt} dt = \frac{1}{x}.$$

Furthermore, the function $x \to \dfrac{1}{x}$ is measurable and Riemann integrable on $[a,b]$, thus

$$\int_{[a,b]} \int_{[0,\infty)} e^{-xt} d\lambda(t)\, d\lambda(x) = \int_{[a,b]} \frac{1}{x}\, d\lambda(x) = \int_a^b \frac{1}{x}\, dx = \log(b/a).$$

On the other hand,

$$\int_{[0,\infty)} \int_{[a,b]} e^{-xt} d\lambda(x)\, d\lambda(t) = \int_{[0,\infty)} \int_a^b e^{-xt} dx\, d\lambda(t) = \int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t).$$

Therefore, $\int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t) = \log(b/a)$.

(5) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and $f \in \mathcal{M}(\mathcal{A})$ satisfies $f^n \in \mathcal{L}^1(\mu)$ for all $n \geq 1$.

(a) Show that if $\lim_{n\to\infty} \int f^n\, d\mu$ exists and is finite, then $|f(x)| \leq 1$ μ a.e. (1 pt)

(b) Show that $\int f^n\, d\mu = c$ is a constant for all $n \geq 1$ if and only if $f = \mathbf{1}_A$ μ a.e. for some measurable set $A \in \mathcal{A}$. (1 pt)

Proof (a) Let $E = \{x \in X : |f(x)| > 1\}$ and assume for the sake of getting a contradiction that $\mu(E) > 0$. For $k \geq 1$, let $E_k = \{x \in X : |f(x) > 1 + 1/k\}$. Then E_k is an increasing sequence of measurable set with $E = \bigcup_{k=1}^{\infty} E_k$. Since $\mu(E) > 0$, then there exists $k \geq 1$ sufficiently large such that $\mu(E_k) > 0$. Note that for any $n \geq 1$,

$$f^{2n} = f^{2n}\mathbf{1}_{E_k} + f^{2n}\mathbf{1}_{E_k^c} \geq f^{2n}\mathbf{1}_{E_k} \geq (1 + 1/k)^{2n}\mathbf{1}_{E_k}.$$

Thus, for all $n \geq 1$

$$\int f^{2n}\, d\mu \geq (1 + 1/k)^{2n}\mu(E_k).$$

This implies that

$$\lim_{n\to\infty} \int f^{2n}\, d\mu \geq \lim_{n\to\infty} (1 + 1/k)^{2n}\mu(E_k) = \infty,$$

contradicting the fact that $\lim_{n\to\infty} \int f^n\, d\mu < \infty$. Thus $\mu(E) = 0$ and $|f(x)| \leq 1$ μ a.e.

Proof (b) If $f = \mathbf{1}_A$ for some measurable set $A \in \mathcal{A}$, then $f^n = \mathbf{1}_A$ for all $n \geq 1$ and hence $\int f^n\, d\mu = \mu(A)$ for all $n \geq 1$.

Conversely, assume $\int f^n\, d\mu = c$ for all $n \geq 1$. Since $\lim_{n\to\infty} \int f^n\, d\mu = c$ exists and is finite, then by part (a), we have that $|f(x)| \leq 1$ μ a.e. Let $A = \{x \in X : f(x) = 1\}$, $B = \{x \in X : f(x) = -1\}$, and $C = \{x \in X : |f(x)| < 1\}$. Since $f \in \mathcal{M}(\mathcal{A})$, then $A, B, C \in \mathcal{A}$, and $f = \mathbf{1}_A f + \mathbf{1}_B f + \mathbf{1}_C f$ and for any $n \geq 1$,

$$c = \int f^n\, d\mu = \mu(A) + (-1)^n\mu(B) + \int_C f^n\, d\mu,$$

as well as

$$c = \lim_{n\to\infty} \int f^n\, d\mu = \lim_{n\to\infty} \left(\mu(A) + (-1)^n\mu(B) + \int_C f^n\, d\mu \right).$$

Note that $\lim_{n\to\infty} \mathbf{1}_C f^n(x) = 0$ for all $x \in X$, and $|\mathbf{1}_C f^n(x)| \leq 1$. Since $\mu(X) < \infty$, then $1 \in \mathcal{L}^1(\mu)$, then by Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_C f^n\, d\mu = \lim_{n\to\infty} \int \mathbf{1}_C f^n(x)\, d\mu = \int \lim_{n\to\infty} \mathbf{1}_C f^n(x)\, d\mu = 0.$$

As a result we have

$$c = \lim_{n\to\infty} \left(\mu(A) + (-1)^n\mu(B) \right).$$

If we take the limit along even n, we get $c = \mu(A) + \mu(B)$, and if we take the limit along odd n, we get $c = \mu(A) - \mu(B)$. This implies that $\mu(B) = 0$, and hence $\mathbf{1}_B f = 0$ μ a.e. Therefore, $c = \mu(A) = \mu(A) + \int_C f^n\, d\mu = 0$ for all $n \geq 1$, and hence $\int_C f^n\, d\mu = 0$ for all $n \geq 1$. In particular, $\int_C f^2\, d\mu = \int \mathbf{1}_C f^2\, d\mu = 0$. Since $\mathbf{1}_C f^2 \geq 0$, this implies that $\mathbf{1}_C f^2 = 0$ μ a.e. and hence $\mathbf{1}_C f = 0$ μ a.e. Thus, $f = \mathbf{1}_A f + \mathbf{1}_B f + \mathbf{1}_C f = \mathbf{1}_A f = \mathbf{1}_A$ μ a.e.

We give also a second short proof: Note that

$$\int f^2(1 - f)^2\, d\mu = \int f^2\, d\mu - 2\int f^3\, d\mu + \int f^4\, d\mu = c - 2c + c = 0.$$

Since $f^2(1 - f)^2 \geq 0$, this implies that $f^2(1 - f)^2 = 0$ μ a.e. implying that f is 0 or 1 μ a.e. equivalently f is μ a.e.equals the indicator function $\mathbf{1}_A$ with $A = \{x \in X : f(x) = 1\}$.