Measure and Integration: Quiz 2014-15

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$, and λ is Lebesgue measure. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \sum_{k=0}^{2^n-1} \frac{3k + 2^n}{2^n} \cdot \mathbf{1}_{[k/2^n,(k+1)/2^n)}(x),\ n \geq 1.$$

   (a) Show that f_n is measurable, and $f_n(x) \leq f_{n+1}(x)$ for all $x \in \mathbb{R}$. (1 pt)

   (b) Show that $\int \sup_{n\geq 1} f_n \, d\lambda = \frac{5}{2}$. (2 pts)

2. Let X be a set, and $\mathcal{C} \subseteq \mathcal{P}(X)$. Consider $\sigma(\mathcal{C})$, the smallest σ-algebra over X containing $\mathcal{C}$, and let $\mathcal{D}$ be the collection of sets $A \in \sigma(\mathcal{C})$ with the property that there exists a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ (depending on A) such that $A \in \sigma(\mathcal{C}_0)$.

   (a) Show that $\mathcal{D}$ is a σ-algebra over X. (2 pts)

   (b) Show that $\mathcal{D} = \sigma(\mathcal{C})$. (1 pt)

3. Let $(X, \mathcal{A}, \mu)$ be a **finite** measure space (so $\mu(X) < \infty$), and $T : X \to X$ an $\mathcal{A}/\mathcal{A}$-measurable function satisfying $\mu(A) = \mu(T^{-1}(A))$ for all $A \in \mathcal{A}$. For $n \geq 1$, denote by $T^n = T \circ T \circ \cdots \circ T$ the n-fold composition of T with itself.

   (a) For $B \in \mathcal{A}$, let $D(B) = \{x \in B : T^n(x) \notin B \text{ for all } n \geq 1\}$. Show that $D(B) \in \mathcal{A}$. (1 pt)

   (b) For $n \geq 1$, let $D(B)_n = T^{-n}(D(B))$. Show that $\mu(D(B)_n) = \mu(D(B))$, for $n \geq 1$, and that $D(B)_n \cap D(B)_m = \emptyset$ if $n \neq m$. (1 pt)

   (c) Show that $\mu(D(B)) = 0$. (1 pt)

   (d) Suppose $A \in \mathcal{A}$ satisfies the property that if $B \in \mathcal{A}$ with $\mu(B) > 0$, then there exists $n \geq 1$ such that $\mu(A \cap T^{-n}B) > 0$. Show that $\mu(A) > 0$, and if additionally $T^{-1}(A) = A$, then $\mu(A) = \mu(X)$. (1 pt)