Measure and Integration: Quiz 2013-14

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$, and λ is Lebesgue measure.
   (a) Show that any monotonically increasing or decreasing function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable i.e. $\mathcal{B}(\mathbb{R}) \setminus \mathcal{B}(\mathbb{R})$ measurable. (1.5 pts)
   (b) Show that for any $f \in \mathcal{M}^+(\mathbb{R})$, and any $a \in \mathbb{R}$, one has
   $$\int_{\mathbb{R}} f(x-a)\, d\lambda(x) = \int_{\mathbb{R}} f(x)\, d\lambda(x).$$
   (Hint: start with simple functions.) (1.5 pts)

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $(X, \mathcal{A}^*, \overline{\mu})$ be its completion (see exercise 4.13, p.29).
   (a) Show that for any $f \in \mathcal{E}^+(\mathcal{A}^*)$, there exists a function $g \in \mathcal{E}^+(\mathcal{A})$ such that $g(x) \le f(x)$ for all $x \in X$, and
   $$\overline{\mu}(\{x \in X : f(x) \neq g(x)\}) = 0.$$
   (1.5 pts)
   (b) Using Theorem 8.8, show that if $u \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A}^*)$, then there exists $w \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$ such that $w(x) \le u(x)$ for all $x \in X$, and
   $$\overline{\mu}(\{x \in X : w(x) \neq u(x)\}) = 0.$$
   (1.5 pts)

3. Let $(X, \mathcal{B}, \mu)$ be a **finite** measure space and $\mathcal{A}$ be a collection of subsets generating $\mathcal{B}$, i.e. $\mathcal{B} = \sigma(\mathcal{A})$, and satisfying the following conditions: (i) $X \in \mathcal{A}$, (ii) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$, and (iii) if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$. Let
   $$\mathcal{D} = \{A \in \mathcal{B} : \forall \varepsilon > 0, \exists C \in \mathcal{A} \text{ such that } \mu(A \Delta C) < \varepsilon\}.$$
   (a) Show that if $(A_n)_n \subset \mathcal{D}$ and $\varepsilon > 0$, then there exists a sequence $(C_n)_n \subset \mathcal{A}$ such that
   $$\mu\left(\bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{\infty} C_n\right) < \varepsilon/2.$$
   (1 pt)
   (b) Use Theorem 4.4 $(iii)'$ to show that there exists an integer $m \ge 1$ such that
   $$\mu\left(\bigcup_{n=1}^{\infty} A_n \Delta \bigcup_{n=1}^{m} C_n\right) < \varepsilon.$$
   (1 pt)
   (c) Show that $\mathcal{D}$ is a σ-algebra. (1 pt)
   (d) Show that $\mathcal{B} = \mathcal{D}$. (1 pt)