Universiteit Utrecht Universiteit Utrecht Boedapestlaan 6

Mathematisch Instituut 3584 CD Utrecht

Hertentamen Maat en Integratie 2012-13

(1) Let $(X, \mathcal{B}, \nu)$ be a measure space, and suppose $X = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\nu(E_n) < \infty$ for all $n \geq 1$. Define μ on $\mathcal{B}$ by $\mu(B) = \sum_{n=1}^{\infty} 2^{-n} \nu(B \cap E_n)/(\nu(E_n) + 1)$.
 (a) Prove that μ is a finite measure on $(X, \mathcal{B})$. (10 pt.)
 (b) Let $B \in \mathcal{B}$. Prove that $\mu(B) = 0$ **if and only if** $\nu(B) = 0$. (10 pt.)

(2) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure. Determine the value of $\lim_{n\to\infty} \int_{(0,n)} x^2 \left(1 - \frac{x}{n}\right)^n d\lambda(x)$. (20 pt.)

(3) Let X be a set, and $\mathcal{C} \subseteq \mathcal{P}(X)$. Consider $\sigma(\mathcal{C})$, the smallest σ-algebra over X containing $\mathcal{C}$, and let $\mathcal{D}$ be the collection of sets $A \in \sigma(\mathcal{C})$ with the property that there exists a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ (depending on A) such that $A \in \sigma(\mathcal{C}_0)$.
 (a) Show that $\mathcal{D}$ is a σ-algebra over X. (12 pt.)
 (b) Show that $\mathcal{D} = \sigma(\mathcal{C})$. (8 pt.)

(4) Let $(X, \mathcal{A}, \mu_1)$ and $(Y, \mathcal{B}, \nu_1)$ be σ-finite measure spaces. Suppose $f \in \mathcal{L}^1(\mu_1)$ and $g \in \mathcal{L}^1(\nu_1)$ are non-negative. Define measures μ_2 on $\mathcal{A}$ and ν_2 on $\mathcal{B}$ by

$$\mu_2(A) = \int_A f \, d\mu_1 \text{ and } \nu_2(B) = \int_B g \, d\nu_1,$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
 (a) For $D \in \mathcal{A} \otimes \mathcal{B}$ and $y \in Y$, let $D_y = \{x \in X : (x, y) \in D\}$. Show that if $\mu_1(D_y) = 0$ ν_1 a.e., then $\mu_2(D_y) = 0$ ν_2 a.e. (7 pt.)
 (b) Show that if $D \in \mathcal{A} \otimes \mathcal{B}$ is such that $(\mu_1 \times \nu_1)(D) = 0$ then $(\mu_2 \times \nu_2)(D) = 0$. (6 pt.)
 (c) Show that for every $D \in \mathcal{A} \otimes \mathcal{B}$ one has

$$(\mu_2 \times \nu_2)(D) = \int_D f(x) g(y) \, d(\mu_1 \times \nu_1)(x, y).$$

 (7 pt.)

(5) Let $(X, \mathcal{A}, \mu)$ be a probability space and let $f \in \mathcal{M}(\mathcal{A})$. Suppose $(f_n) \subset \mathcal{M}(\mathcal{A})$ converges in measure to f, i.e. $f_n \xrightarrow{\mu} f$.
 (a) Show that there exists a sequence $n_1 < n_2 < \cdots$ such that

$$\mu(\{x \in X : |f_{n_k}(x) - f(x)| > 1/k\}) \leq 2^{-k},$$

 for all $k \geq 1$. (8 pt.)
 (b) Let $A_k = \{x \in X : |f_{n_k}(x) - f(x)| > 1/k\}$ and $A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Show that $\mu(A) = 0$, and $\lim_{n\to\infty} f_{n_k}(x) = f(x)$ for all $x \notin A$. Conclude that $f_{n_k} \to f$ μ a.e. (12 pt.)