Universiteit Utrecht Boedapestlaan 6

Mathematisch Instituut 3584 CD Utrecht

Measure and Integration: Extra Retake Final 2015-16

(1) Consider the measure space $([0,1]\mathcal{B}([0,1]),\lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra to $[0,1]$, and λ is the restriction of Lebesgue measure to $[0,1]$. Let $E_1,\cdots,E_m$ be a collection of Borel measurable subsets of $[0,1]$ such that every element $x\in[0,1]$ belongs to at least n sets in the collection $\{E_j\}_{j=1}^m$, where $n\leq m$. Show that there exists a $j\in\{1,\cdots,m\}$ such that $\lambda(E_j)\geq\dfrac{n}{m}$. (1 pt)

(2) Let $(X,\mathcal{F},\mu)$ be a measure space, and $1<p,q<\infty$ conjugate numbers, i.e. $1/p+1/q=1$. Show that if $f\in\mathcal{L}^p(\mu)$, then there exists $g\in\mathcal{L}^q(\mu)$ such that $||g||_q=1$ and $\int fg\,d\mu=||f||_p$. (1 pt)

(3) Let $(X,\mathcal{A})$ be a measurable space and μ,ν are finite measure on $\mathcal{A}$. Show that there exists a function $f\in\mathcal{L}^1_+(\mu)\cap\mathcal{L}^1_+(\nu)$ such that for every $A\in\mathcal{A}$, we have

$$\int_A(1-f)\,d\mu=\int_A f\,d\nu.$$

(2 pts)

(4) Consider the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure. Determine the value of

$$\lim_{n\to\infty}\int_{(0,n)}(1+\frac{x}{n})^{-n}(1-\sin\frac{x}{n})\,d\lambda(x).$$

(2 pts)

(5) Let $E_1=E_2=\mathbb{N}=\{1,2,3,\cdots\}$. Let $\mathcal{B}$ be the collection of all subsets of $\mathbb{N}$. and $\mu_1=\mu_2$ be counting measure on $\mathbb{N}$. Let $f:E_1\times E_2\to\mathbb{R}$ by $f(n,n)=n$, $f(n,n+1)=-n$ and $f(n,m)=0$ for $m\neq n,n+1$.
   (a) Prove that $\int_{E_1}\int_{E_2}f(n,m)d\mu_2(m)d\mu_1(n)=0$. (0.75 pt)
   (b) Prove that $\int_{E_2}\int_{E_1}f(n,m)d\mu_1(n)d\mu_2(m)=\infty$. (0.75 pt)
   (c) Explain why parts (a) and (b) do not contradict Fubini's Theorem. (0.5)

(6) Let $(X,\mathcal{A},\mu)$ be a σ-finite measure space, and (f_j) a uniformly integrable sequence of measurable functions. Define $F_k=\sup_{1\leq j\leq k}|f_j|$ for $k\geq 1$.
   (a) Show that for any $w\in\mathcal{M}^+(\mathcal{A})$,

$$\int_{\{F_k>w\}}F_k\,d\mu\leq\sum_{j=1}^k\int_{\{|f_j|>w\}}|f_j|\,d\mu.$$

   (0.5)
   (b) Show that for every $\epsilon>0$, there exists a $w_\epsilon\in\mathcal{L}^1_+(\mu)$ such that for all $k\geq 1$

$$\int_X F_k\,d\mu\leq\int_X w_\epsilon\,d\mu+k\epsilon.$$

   (1 pt)
   (c) Show that

$$\lim_{k\to\infty}\frac{1}{k}\int_X F_k\,d\mu=0.$$

   (0.5 pt)