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## Measure and Integration: Extra Retake Final 2015-16

(1) Consider the measure space $([0,1]\mathcal{B}([0,1]),\lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel $\sigma$-algebra to $[0,1]$, and $\lambda$ is the restriction of Lebesgue measure to $[0,1]$. Let $E_1,\cdots,E_m$ be a collection of Borel measurable subsets of $[0,1]$ such that every element $x\in[0,1]$ belongs to at least $n$ sets in the collection $\{E_j\}_{j=1}^m$, where $n\leq m$. Show that there exists a $j\in\{1,\cdots,m\}$ such that $\lambda(E_j)\geq\dfrac{n}{m}$. (1 pt)

**Solution**: By hypothesis, for any $x\in[0,1]$ we have $\displaystyle\sum_{j=}^{m}\mathbf{1}_{E_j}(x)\geq n$. Assume for the sake of getting a contradiction that $\lambda(E_j)<\dfrac{n}{m}$ for all $1\leq j\leq m$. Then,

$$n=\int_{[0,1]}n\,d\lambda\leq\int\sum_{j=}^{m}\mathbf{1}_{E_j}(x)\,d\lambda=\sum_{j=1}^{m}\lambda(E_j)<\sum_{j=}^{m}\frac{n}{m}=n,$$

a contradiction. Hence, there exists $j\in\{1,\cdots,m\}$ such that $\lambda(E_j)\geq\dfrac{n}{m}$.

(2) Let $(X,\mathcal{F},\mu)$ be a measure space, and $1<p,q<\infty$ conjugate numbers, i.e. $1/p+1/q=1$. Show that if $f\in\mathcal{L}^p(\mu)$, then there exists $g\in\mathcal{L}^q(\mu)$ such that $||g||_q=1$ and $\int fg\,d\mu=||f||_p$. (1 pt)

**Solution**: Note that $q(p-1)=p$, so we define $g=\text{sgn}(f)\left(\dfrac{f}{||f||_p}\right)^{p-1}$. Then,

$$\int|g|^q\,d\mu=\int\frac{|f|^p}{||f||_p^p}\,d\mu=1.$$

So $||g||_q=1$ and

$$\int fg\,d\mu=\int|fg|\,d\mu=\int\frac{|f|^p}{||f||_p^{p-1}}\,d\mu=||f||_p.$$

(3) Let $(X,\mathcal{A})$ be a measurable space and $\mu,\nu$ are finite measure on $\mathcal{A}$. Show that there exists a function $f\in\mathcal{L}^1_+(\mu)\cap\mathcal{L}^1_+(\nu)$ such that for every $A\in\mathcal{A}$, we have

$$\int_A(1-f)\,d\mu=\int_A f\,d\nu.$$

(2 pts)

**Proof**: First note that $\mu+\nu$ is a measure (Exercise 4.6(ii)), and that $\mu\ll\mu+\nu$. By using a standard argument (first checking indictor functions, then simple functions, then positive functions, then general integrable functions) one sees that for any $g\in\mathcal{L}^1(\mu+\nu)$ one has $g\in\mathcal{L}^1(\mu)\cap\mathcal{L}^1(\nu)$, and

$$\int g\,d(\mu+\nu)=\int g\,d\mu+\int g\,d\nu.$$

Now the condition $\int_A(1-f)\,d\mu=\int_A f\,d\nu$ is equivalent to $\mu(A)=\int_A f\,d(\mu+\nu)$. Since $\mu\ll\mu+\nu$, then by Radon-Nikodym Theorem there exists $f\in\mathcal{L}^1_+(\mu+\nu)$ such that $\mu(A)=\int_A f\,d(\mu+\nu)$. Thus, $f\in\mathcal{L}^1_+(\mu)\cap\mathcal{L}^1_+(\nu)$ and $\int_A(1-f)\,d\mu=\int_A f\,d\nu$ for all $A\in\mathcal{A}$.

(4) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra, and $\lambda$ Lebesgue measure. Determine the value of

$$\lim_{n\to\infty} \int_{(0,n)} (1+\frac{x}{n})^{-n}(1-\sin\frac{x}{n})\, d\lambda(x).$$

(2 pts)

**Solution**: Let $u_n(x) = \mathbf{1}_{(0,n)}(1+\frac{x}{n})^{-n}(1-\sin\frac{x}{n})$. The positive sequence $\left((1+\frac{x}{n})^{-n}\right)_n$ decreases to $e^{-x}\mathbf{1}_{(0,\infty)}$ and the sequence $\left(1-\sin\frac{x}{n}\right)_n$ is bounded from below by 0 and from above by 2 and converges to 1 as $n\to\infty$. Thus, $\lim_{n\to\infty} u_n(x) = \mathbf{1}_{(0,\infty)}e^{-x}$, and $0 \le u_n(x) \le 2(1+\frac{x}{2})^{-2}\mathbf{1}_{(0,\infty)}$ for $n\ge 2$ and all $x\in\mathbb{R}$. Since the function $2(1+\frac{x}{2})^{-2}\mathbf{1}_{(0,\infty)}$ is measurable, non-negative and the improper Riemann integrable on $(0,\infty)$ exists, it follows that it is Lebesgue integrable on $(0,\infty)$. By Lebesgue Dominated Convergence Theorem (and taking the limit for $n\ge 2$), we have

$$\begin{aligned}\lim_{n\to\infty} \int_{(0,n)} (1+\frac{x}{n})^{-n}(1-\sin\frac{x}{n})\, d\lambda(x) &= \lim_{n\to\infty}\int u_n(x)d\lambda(x)\\ &= \int \mathbf{1}_{(0,\infty)}e^{-x}\, d\lambda(x) = \int_0^\infty e^{-x}\,dx = 1.\end{aligned}$$

(5) Let $E_1 = E_2 = \mathbb{N} = \{1,2,3,\cdots\}$. Let $\mathcal{B}$ be the collection of all subsets of $\mathbb{N}$. and $\mu_1 = \mu_2$ be counting measure on $\mathbb{N}$. Let $f : E_1\times E_2 \to \mathbb{R}$ by $f(n,n) = n$, $f(n,n+1) = -n$ and $f(n,m) = 0$ for $m \ne n, n+1$.

(a) Prove that $\int_{E_1}\int_{E_2} f(n,m)d\mu_2(m)d\mu_1(n) = 0$. (0.75 pt)

(b) Prove that $\int_{E_2}\int_{E_1} f(n,m)d\mu_1(n)d\mu_2(m) = \infty$. (0.75 pt)

(c) Explain why parts (a) and (b) do not contradict Fubini's Theorem. (0.5)

**Proof (a)** For each fixed $n$ one has

$$\int_{E_2} f(n,m)d\mu_2(m) = f(n,n)\mu_2(\{n\}) + f(n,n+1)\mu_2(\{n+1\}) = 0.$$

Thus, $\int_{E_1}\int_{E_2} f(n,m)d\mu_2(m)d\mu_1(n) = 0$.

**Proof (b)** For each fixed $m$,

$$\int_{E_1} f(n,m)d\mu_1(n) = f(m,m)\mu_1(\{m\}) + f(m-1,m)\mu_1(\{m-1\}) = 1.$$

Thus, $\int_{E_2}\int_{E_1} f(n,m)d\mu_1(n)d\mu_2(m) = \int_{E_2} 1 d\mu_2(m) = \mu_2(E_2) = \infty$.

**Proof (c)** Parts (a) and (b) do not contradict Fubini's Theorem because the function $f$ is not $\mu_1\times\mu_2$ integrable. This follows from

$$\int_{E_1}\int_{E_2} |f(n,m)|d\mu_2(m)d\mu_1(n) = \int_{E_1} 2n d\mu_1(n) = \sum_{n=1}^\infty 2n = \infty.$$

(6) Let $(X,\mathcal{A},\mu)$ be a $\sigma$-finite measure space, and $(f_j)$ a uniformly integrable sequence of measurable functions. Define $F_k = \sup_{1\le j\le k}|f_j|$ for $k\ge 1$.

(a) Show that for any $w\in\mathcal{M}^+(\mathcal{A})$,

$$\int_{\{F_k>w\}} F_k\,d\mu \le \sum_{j=1}^k \int_{\{|f_j|>w\}} |f_j|\,d\mu.$$

(0.5)

(b) Show that for every $\epsilon>0$, there exists a $w_\epsilon\in\mathcal{L}^1_+(\mu)$ such that for all $k\ge 1$

$$\int_X F_k\,d\mu \le \int_X w_\epsilon\,d\mu + k\epsilon.$$

(1 pt)

(c) Show that
$$\lim_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu = 0.$$
(0.5 pt)

**Proof (a)** Let $w \in \mathcal{M}^+(\mathcal{A})$, then
$$\begin{aligned} \int_{\{F_k > w\}} F_k \, d\mu &\leq \sum_{j=1}^{k} \int_{\{F_k > w\} \cap \{|f_j| = F_k\}} F_k \, d\mu \\ &\leq \sum_{j=1}^{k} \int_{\{|f_j| > w\}} |f_j.| \, d\mu. \end{aligned}$$

**Proof (b)** Let $\epsilon > 0$. By uniform integrability of the sequence $(f_j)$ there exists $w_\epsilon \in \mathcal{L}^+(\mu)$ such that
$$\int_{\{|f_j| > w_\epsilon\}} |f_j| \, d\mu < \epsilon$$
for all $j \geq 1$. By part (a)
$$\int_{\{F_k > w_\epsilon\}} F_k \, d\mu \leq \sum_{j=1}^{k} \int_{\{|f_j| > w_\epsilon\}} |f_j| \, d\mu \leq k\epsilon.$$
Now,
$$\begin{aligned} \int_X F_k \, d\mu &= \int_{\{F_k > w_\epsilon\}} F_k \, d\mu + \int_{\{F_k \leq w_\epsilon\}} F_k \, d\mu \\ &\leq k\epsilon + \int_X w_\epsilon \, d\mu. \end{aligned}$$

**Proof (c)** For any $\epsilon > 0$, by part (b),
$$\frac{1}{k} \int_X F_k \, d\mu \leq \frac{1}{k} \int_X w_\epsilon \, d\mu + \epsilon.$$
Thus,
$$\limsup_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu \leq \epsilon,$$
for any $\epsilon$. Since $F_k \geq 0$, we see that
$$\limsup_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu = \lim_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu = 0.$$