Measure and Integration Quiz Extra, 2016-17

1. Let X be a set and $\mathcal{F}$ a collection of real valued functions on X satisfying the following properties:

   (i) $\mathcal{F}$ contains the constant functions,

   (ii) if $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$, then $f+g, fg, cf \in \mathcal{F}$,

   (ii) if $f_n \in \mathcal{F}$, and $f = \lim_{n\to\infty} f_n$, then $f \in \mathcal{F}$.

   For $A \subseteq X$, denote by $\mathbf{1}_A$ the indicator function of A, i.e.

   $$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A, \\ 0, & x \notin A. \end{cases}$$

   Show that the collection $\mathcal{A} = \{A \subseteq X : \mathbf{1}_A \in \mathcal{F}\}$ is a σ-algebra.

2. Let $(X, \mathcal{D}, \mu)$ be a measure space, and let $\overline{\mathcal{D}}^\mu$ be the completion of the σ-algebra $\mathcal{D}$ with respect to the measure μ (see exercise 4.13, p.29). We denote by $\overline{\mu}$ the extension of the measure μ to the σ-algebra $\overline{\mathcal{D}}^\mu$. Suppose $f : X \to X$ is a function such that $f^{-1}(B) \in \mathcal{D}$ and $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{D}$. Show that $f^{-1}(\overline{B}) \in \overline{\mathcal{D}}^\mu$ and $\overline{\mu}(f^{-1}(\overline{B})) = \overline{\mu}(\overline{B})$ for all $\overline{B} \in \overline{\mathcal{D}}^\mu$.

3. Consider the measure space $([0,1]\mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra to $[0,1]$, and λ is the restriction of Lebesgue measure to $[0,1]$. Let $E_1, \cdots, E_m$ be a collection of Borel measurable subsets of $[0,1]$ such that every element $x \in [0,1]$ belongs to at least n sets in the collection $\{E_j\}_{j=1}^m$, where $n \leq m$. Show that there exists a $j \in \{1, \cdots, m\}$ such that $\lambda(E_j) \geq \frac{n}{m}$.

4. Let μ and ν be two measures on the measure space $(E, \mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$. Show that if f is any non-negative measurable function on $(E, \mathcal{B})$, then $\int_E f\, d\mu \leq \int_E f\, d\nu$.