Measure and Integration: Final 2016-17

(1) Consider the measure space $[1,\infty),\mathcal{B}([1,\infty]),\lambda)$ where $\mathcal{B}([1,\infty])$ is the Borel σ-algebra and λ is the Lebesgue measure restricted to $[1,\infty)$. Show that

$$\lim_{n\to\infty}\int_{[1,\infty)}\frac{n\sin(x/n)}{x^3}\,d\lambda(x)=1.$$

(Hint: $\lim\limits_{x\to 0}\sin(x)/x=1$) (2 pts)

(2) Let $(X,\mathcal{A},\mu)$ be a finite measure space, and $\Phi:[0,\infty)\to[0,\infty)$ a monotonically increasing function such that $\lim\limits_{r\to\infty}\dfrac{\Phi(r)}{r}=\infty$. Let $M>0$, and

$$\mathcal{F}=\{f\in\mathcal{L}^1(\mu):\int_X\Phi\circ|f|\,d\mu\le M\}.$$

(a) Prove that for each $\epsilon>0$, there exists a real number $N>0$ such that for all $f\in\mathcal{F}$ one has

$$\int_{\{|f|>N\}}|f|\,d\mu\le\frac{\epsilon}{M}\int_{\{|f|>N\}}\Phi\circ|f|\,d\mu.$$

(1 pt)

(b) Let $1\le p<\infty$ and (f_n) be a sequence of measurable functions such that $f_n^p\in\mathcal{F}$ for $n\ge 1$. Assume that $f_n\xrightarrow{\mu}f$ i.e. (f_n) converges to f in μ measure with $f\in\mathcal{L}^p(\mu)$. Show that $\lim\limits_{n\to\infty}||f_n-f||_p=0$. (1 pt)

(3) Consider the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ is Lebesgue measure.

(a) Prove that for $f\in\mathcal{L}^1(\lambda)$, and $n\in\mathbb{Z}$ one has $\displaystyle\int_{[0,1]}f(x+n)\,d\lambda(x)=\int_{[n,n+1]}f(x)\,d\lambda(x)$.

(1.5 pts)

(b) Let $f\in\mathcal{L}^1(\lambda)$, and define $g(x)=\mathbf{1}_{[0,1]}(x)\displaystyle\sum_{n\in\mathbb{Z}}f(x+n)$. Show that $g\in\mathcal{L}^1(\lambda)$ and that

$$\int_{\mathbb{R}}g(x)\,d\lambda(x)=\int_{\mathbb{R}}f(x)\,d\lambda(x).$$

(1 pt)

(4) Let $(X,\mathcal{A},\mu)$ be a measure space, and $p,q\in(1,\infty)$ and $r\ge 1$ be such that $1/r=1/p+1/q$. Show that if $f\in\mathcal{L}^p(\mu)$ and $g\in\mathcal{L}^q(\mu)$, then $fg\in\mathcal{L}^r(\mu)$ and $||fg||_r\le||f||_p\,||g||_q$. (1.5 pts)

(5) Let $E=\{(x,y):0<x<1,\,0<y<\infty\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda\times\lambda$. Let $f:E\to\mathbb{R}$ be given by $f(x,y)=e^{-y}\sin(2xy)$.

(a) Show that f is $\lambda\times\lambda$ integrable on E. (0.5 pts)

(b) Applying Fubini's Theorem to the function f, show that

$$\int_0^\infty e^{-y}\frac{\sin^2(y)}{y}\,d\lambda(y)=\frac{\log 5}{4}.$$

(Hint: use integration by parts twice to calculate $(R)\displaystyle\int_0^\infty e^{-y}\sin(2xy)\,dy$) (1.5 pts)