Measure and Integration: Solution Final 2016-17

(1) Consider the measure space $[1,\infty),\mathcal{B}([1,\infty]),\lambda)$ where $\mathcal{B}([1,\infty])$ is the Borel σ-algebra and λ is the Lebesgue measure restricted to $[1,\infty)$. Show that

$$\lim_{n\to\infty}\int_{[1,\infty)}\frac{n\sin(x/n)}{x^3}\,d\lambda(x)=1.$$

(Hint: $\lim_{x\to 0}\sin(x)/x=1$) (2 pts)

Proof: Let $u_n(x)=\dfrac{n\sin(x/n)}{x^3}$, then u_n is continuous on $[1,\infty)$ and hence is measurable. Note that $|\sin(y)|\leq y$ for all $y\geq 0$, hence $u_n(x)\leq 1/x^2$. Furthermore, $\lim_{n\to\infty}u_n(x)=\dfrac{1}{x^2}$. Since the function $\dfrac{1}{x^2}$ is positive, measurable and the improper Riemann integral on $[1,\infty)$ exists, it follows it is Lebesgue integrable on $[1,\infty)$. By Lebesgue Dominated Convergence Theorem we have

$$\begin{aligned}\lim_{n\to\infty}\int_{[1,\infty)}\frac{n\sin(x/n)}{x^3}\,d\lambda(x) &= \int_{[1,\infty)}\lim_{n\to\infty}\frac{n\sin(x/n)}{x^3}\,d\lambda(x)\\ &= \int_{[1,\infty)}\frac{1}{x^2}\,d\lambda(x)\\ &= (R)\int_1^\infty\frac{1}{x^2}\,dx=1.\end{aligned}$$

(2) Let $(X,\mathcal{A},\mu)$ be a finite measure space, and $\Phi:[0,\infty)\to[0,\infty)$ a monotonically increasing function such that $\lim_{r\to\infty}\dfrac{\Phi(r)}{r}=\infty$. Let $M>0$, and

$$\mathcal{F}=\{f\in\mathcal{L}^1(\mu):\int_X\Phi\circ|f|\,d\mu\leq M\}.$$

(a) Prove that for each $\epsilon>0$, there exists a real number $N>0$ such that for all $f\in\mathcal{F}$ one has

$$\int_{\{|f|>N\}}|f|\,d\mu\leq\frac{\epsilon}{M}\int_{\{|f|>N\}}\Phi\circ|f|\,d\mu.$$

(1 pt)

(b) Let $1\leq p<\infty$ and (f_n) be a sequence of measurable functions such that $f_n^p\in\mathcal{F}$ for $n\geq 1$. Assume that $f_n\xrightarrow{\mu}f$ i.e. (f_n) converges to f in μ measure with $f\in\mathcal{L}^p(\mu)$. Show that $\lim_{n\to\infty}||f_n-f||_p=0$. (1 pt)

Proof (a): First note by Exercise 2(a) of hand-in set 3 that Φ is Borel measurable. Let $\epsilon>0$, since $\lim_{r\to\infty}\dfrac{\Phi(r)}{r}=\infty$ one can find $N>0$ such that $\dfrac{\Phi(r)}{r}\geq\dfrac{M}{\epsilon}$ for all $r>N$. Hence, if $f\in\mathcal{F}$, and $x\in\{x\in X:|f|(x)>N\}$ then $\dfrac{\Phi(|f|(x))}{|f|(x)}\geq\dfrac{M}{\epsilon}$, i.e. $|f|(x)\leq\dfrac{\epsilon}{M}\Phi(|f|(x))$. Thus,

$$\int_{\{|f|>N\}}|f|\,d\mu\leq\frac{\epsilon}{M}\int_{\{|f|>N\}}\Phi\circ|f|\,d\mu.$$

Proof (b): We first show that the collection $\mathcal{F}$ is uniformly integrable. Let $\epsilon > 0$, and N as in part (a). Since $\mu(X) < \infty$, then the constant function $w_\epsilon(x) = N$ is in $\mathcal{L}^1(\mu)$. Since $\Phi \circ |f| \geq 0$, by part (a) we have for any $f \in \mathcal{F}$,

$$\int_{\{|f|>N\}} |f|\, d\mu \leq \frac{\epsilon}{M} \int_{\{|f|>N\}} \Phi \circ |f|\, d\mu \leq \frac{\epsilon}{M} \int_X \Phi \circ |f|\, d\mu \leq \epsilon.$$

Hence, the collection $\mathcal{F}$ is uniformly integrable. Since the function $x \to x^p$ is continuous, by Exercise 6.10 (iii) we see that $f_n^p \xrightarrow{\mu} f^p$. Since $f_n^p \in \mathcal{F}$, the sequence $(|f_n|^p)$ is uniformly integrable, hence by Vitali's Theorem $\lim\limits_{n\to\infty} ||f_n - f||_p = 0$.

(3) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ is Lebesgue measure.

(a) Prove that for $f \in \mathcal{L}^1(\lambda)$, and $n \in \mathbb{Z}$ one has $\displaystyle\int_{[0,1]} f(x+n)\, d\lambda(x) = \int_{[n,n+1]} f(x)\, d\lambda(x)$. (1.5 pts)

(b) Let $f \in \mathcal{L}^1(\lambda)$, and define $g(x) = \mathbf{1}_{[0,1]}(x) \sum\limits_{n\in\mathbb{Z}} f(x+n)$. Show that $g \in \mathcal{L}^1(\lambda)$ and that

$$\int_{\mathbb{R}} g(x)\, d\lambda(x) = \int_{\mathbb{R}} f(x)\, d\lambda(x).$$

(1 pt)

Proof (a): We use the standard argument. First assume $f = \mathbf{1}_A$ where $A \in \mathcal{B}(\mathbb{R})$. Note that for any $x \in R$ and $n \in Z$, we have $\mathbf{1}_A(x+n) = \mathbf{1}_{A-n}(x)$. Since λ is translation invariant, then $\lambda\Big([0,1] \cap (A-n)\Big) = \lambda\Big([n, n+1] \cap A\Big)$. Now,

$$\int_{[0,1]} \mathbf{1}_A(x+n)\, d\lambda(x) = \int_{\mathbb{R}} \mathbf{1}_{[0,1]\cap(A-n)}(d)\, d\lambda(x) = \lambda\Big([0,1] \cap (A-n)\Big),$$

and

$$\int_{[n,n+1]} \mathbf{1}_A(x)\, d\lambda(x) = \int_{\mathbb{R}} \mathbf{1}_{[n,n+1]\cap A}\, d\lambda(x) = \lambda\Big([n, n+1] \cap A\Big).$$

Thus, $\displaystyle\int_{[0,1]} \mathbf{1}_A(x+n)\, d\lambda(x) = \int_{[n,n+1]} \mathbf{1}_A(x)\, d\lambda(x)$. Assume now that $f = \sum\limits_{i=0}^{n} a_i \mathbf{1}_{A_i}$ be a non-negative measurable simple function (so $A_i \in \mathcal{B}(\mathbb{R})$), using the linearity of the integral, we have

$$\begin{aligned}
\int_{[0,1]} f(x+n)\, d\lambda(x) &= \int_{[0,1]} \sum_{i=0}^{n} a_i \mathbf{1}_{A_i}(x+n)\, d\lambda(x) \\
&= \sum_{i=0}^{n} a_i \int_{[0,1]} \mathbf{1}_{A_i}(x+n)\, d\lambda(x) \\
&= \sum_{i=0}^{n} a_i \int_{[n,n+1]} \mathbf{1}_{A_i}(x)\, d\lambda(x) \\
&= \int_{[n,n+1]} \sum_{i=0}^{n} a_i \mathbf{1}_{A_i}(x)\, d\lambda(x) \\
&= \int_{[n,n+1]} f(x)\, d\lambda(x).
\end{aligned}$$

Now, assume f is a non-negative integrable function. Then there exists an increasing sequence (f_m) of non-negative simple functions with $f_m \nearrow f$ (pointwise). Thus, for eaxh $x \in \mathbb{R}$ and $n \in Z$ we have $f_m(x+n) \nearrow f(x+n)$, $\mathbf{1}_{[0,1]}(x) f_m(x+n) \nearrow \mathbf{1}_{[0,1]}(x) f(x+n)$, and $\mathbf{1}_{[n,n+1]}(x) f_m(x) \nearrow$

$\mathbf{1}_{[n,n+1]}(x)f(x)$. By Beppo-Lévy we have

$$\begin{aligned}\int_{[0,1]} f(x+n)\,d\lambda(x) &= \sup_m \int_{[0,1]} f_m(x+n)\,d\lambda(x)\\ &= \sup_m \int_{[n,n+1]} f_m(x)\,d\lambda(x)\\ &= \int_{[n,n+1]} f(x)\,d\lambda(x).\end{aligned}$$

Finally, assume $f \in \mathcal{L}^1(\lambda)$, then f^+, f^- are non-negative integrable functions,hence

$$\begin{aligned}\int_{[0,1]} f(x+n)\,d\lambda(x) &= \int_{[0,1]} f^+(x+n)\,d\lambda(x) - \int_{[0,1]} f^-(x+n)\,d\lambda(x)\\ &= \int_{[n,n+1]} f^+(x)\,d\lambda(x) - \int_{[n,n+1]} f^-(x)\,d\lambda(x)\\ &= \int_{[n,n+1]} f(x)\,d\lambda(x).\end{aligned}$$

Proof (b): Using Corollary 9.9 and part (a), we have

$$\begin{aligned}\int_{\mathbb{R}} |g(x)|\,d\lambda(x) &= \int_{[0,1]} \Big|\sum_{n\in Z} f(x+n)\Big|\,d\lambda\\ &\leq \int_{[0,1]} \sum_{n\in Z} |f(x+n)|\,d\lambda\\ &= \sum_{n\in Z}\int_{[0,1]} |f(x+n)|\,d\lambda\\ &= \sum_{n\in Z}\int_{[n,n+1]} |f(x)|\,d\lambda\\ &= \int_{\mathbb{R}} |f(x)|\,d\lambda < \infty.\end{aligned}$$

Hence $g \in \mathcal{L}^1(\lambda)$. Note that the above shows that the series $g(x) = \mathbf{1}_{[0,1]}(x)\sum_{n\in\mathbb{Z}} f(x+n)$ is absolutely convergent λ a.e. and that $\sum_{n\in Z}\int_{[0,1]} |f(x+n)|\,d\lambda = \int_{\mathbb{R}} |f(x)|\,d\lambda < \infty$. Hence by Exercise 11.4 (i.e. applying Lebesgue Dominated Convergence Theorem) we have

$$\begin{aligned}\int_{\mathbb{R}} g(x)\,d\lambda(x) &= \int_{[0,1]} g(x)\,d\lambda(x)\\ &= \sum_{n\in Z}\int_{[0,1]} f(x+n)\,d\lambda(x)\\ &= \sum_{n\in Z}\int_{[n,n+1]} f(x)\,d\lambda(x)\\ &= \int_{\mathbb{R}} f(x)\,d\lambda(x).\end{aligned}$$

(4) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $p, q \in (1, \infty)$ and $r \geq 1$ be such that $1/r = 1/p + 1/q$. Show that if $f \in \mathcal{L}^p(\mu)$ and $g \in \mathcal{L}^q(\mu)$, then $fg \in \mathcal{L}^r(\mu)$ and $||fg||_r \leq ||f||_p\,||g||_q$. (1.5 pts)

Proof: Let $p' = p/r$ and $q' = q/r$, since $1/r = 1/p + 1/q$ we have $1 = 1/p' + 1/q'$. Suppose $f \in \mathcal{L}^p(\mu)$ and $g \in \mathcal{L}^q(\mu)$, and set $F = f^r$ and $G = g^r$. Then,

$$\int |F|^{p'}\,d\mu = \int |f|^p\,d\mu < \infty,$$

and

$$\int |G|^{q'}\, d\mu = \int |g|^q\, d\mu < \infty.$$

Hence, $F \in \mathcal{L}^{p'}(\mu)$, and $G \in \mathcal{L}^{q'}(\mu)$. By Hölder's inequality we have $(fg)^r = FG \in \mathcal{L}^1(\mu)$, which implies $fg \in \mathcal{L}^r(\mu)$, and

$$\begin{aligned} \int |fg|^r\, d\mu &= \int |FG|\, d\mu \\ &\leq \Big(\int |F|^{p'}\, d\mu\Big)^{1/p'} \Big(\int |G|^{q'}\, d\mu\Big)^{1/q'} \\ &= \Big(\int |f|^p\, d\mu\Big)^{r/p} \Big(\int |g|^q\, d\mu\Big)^{r/q}. \end{aligned}$$

Hence,

$$\Big(\int |fg|^r\, d\mu\Big)^{1/r} \leq \Big(\int |f|^p\, d\mu\Big)^{1/p} \Big(\int |g|^q\, d\mu\Big)^{1/q},$$

equivalently, $||fg||_r \leq ||f||_p ||g||_q$.

(5) Let $E = \{(x, y) : 0 < x < 1,\ 0 < y < \infty\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x, y) = e^{-y} \sin(2xy)$.
  (a) Show that f is $\lambda \times \lambda$ integrable on E. (0.5 pts)
  (b) Applying Fubini's Theorem to the function f, show that

$$\int_0^\infty e^{-y} \frac{\sin^2(y)}{y}\, d\lambda(y) = \frac{\log 5}{4}.$$

(Hint: use integration by parts twice to calculate $(R) \int_0^\infty e^{-y} \sin(2xy)\, dy)$ (1.5 pts)

Proof(a) Notice that f is continuous, and hence measurable. Furthermore, $|f(x, y)| \leq e^{-y}$. The function $g(y) = e^{-y}$ is non-negative measurable, and the improper Riemann integral exists and is finite. Hence, $\int_{(0,\infty)} e^{-y}\, d\lambda(y) = (R) \int_0^\infty e^{-y}\, dy = 1$. By Tonelli's Theorem

$$\begin{aligned} \int_E |f(x, y)| d(\lambda \times \lambda)(x, y) &\leq \int_E e^{-y} d(\lambda \times \lambda)(x, y) \\ &= \int_0^1 \int_0^\infty e^{-y}\, d\lambda(y)\, d\lambda(x) \\ &= \int_0^1 1\, d\lambda(x) \\ &= 1 < \infty. \end{aligned}$$

This shows that f is $\lambda \times \lambda$ integrable on E.

Proof(b) By Fubini's Theorem,

$$\int_E f(x, y)\, d(\lambda \times \lambda) = \int_{(0,1)} \int_{(0,\infty)} e^{-y} \sin(2xy)\, d\lambda(y)\, d\lambda(x) = \int_{(0,\infty)} \int_{(0,1)} e^{-y} \sin(2xy)\, d\lambda(x)\, d\lambda(y).$$

An easy calculation shows that the Riemann integral

$$(R) \int_0^1 e^{-y} \sin(2xy)\, dx = e^{-y} \frac{1 - \cos(2y)}{2y} = e^{-y} \frac{\sin^2(y)}{y}.$$

We now show that $(R) \int_0^\infty e^{-y} \sin(2xy)\, dy$ exists and is finite. This is done by first integrating twice by parts to get

$$(R) \int_0^\infty e^{-y} \sin(2xy)\, dy = 2x - 4x^2 (R) \int_0^\infty e^{-y} \sin(2xy)\, dy,$$

and hence $(R)\displaystyle\int_0^\infty e^{-y}\sin(2xy)\,dy = \frac{2x}{1+4x^2}$. This shows that

$$\int_{(0,\infty)} e^{-y}\sin(2xy)\,d\lambda(y) = (R)\int_0^\infty e^{-y}\sin(2xy)\,dy = \frac{2x}{1+4x^2}.$$

Since the function $\dfrac{2x}{1+4x^2}$ is continuous on $[0,1]$, the Rimann integral equals the Lebesgue integral, hence

$$\int_{(0,1)} \frac{2x}{1+4x^2}\,d\lambda(x) = \int_{[0,1]} \frac{2x}{1+4x^2}\,d\lambda(x) = (R)\int_0^1 \frac{2x}{1+4x^2}\,dx = \frac{\log 5}{4}.$$

From the above we have

$$\int_0^\infty e^{-y}\frac{\sin^2(y)}{y}\,d\lambda(y) = \int_E f(x,y)\,d(\lambda\times\lambda) = \frac{\log 5}{4}.$$