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Measure and Integration: Mid-Term, 2020-21

(1) Let X be a set and μ, ν two **outer measures** on X, i.e. $\mu, \nu : \mathcal{P}(X) \to [0, \infty]$ satisfying the three properties:

(i) $\mu(\varnothing) = \nu(\varnothing) = 0$,

(ii) if $A, B \in \mathcal{P}(X)$ with $A \subseteq B$, then $\mu(A) \leq \mu(B)$ and $\nu(A) \leq \nu(B)$ (μ and ν are monotone),

(iii) if (A_n) is a sequence in $\mathcal{P}(X)$, then $\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$ and $\nu(\bigcup_n A_n) \leq \sum_n \nu(A_n)$.

Define $\rho : \mathcal{P}(X) \to [0, \infty]$ by $\rho(A) = \max(\mu(A), \nu(A))$. Show that ρ is an outer measure on X, i.e. satisfies properties (i), (ii) and (iii). (2 pts)

(2) Consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the Borel σ-algebra restricted to $[0,1]$ and λ is the restriction of Lebesgue measure on $[0,1]$. Define a map $u : [0,1] \to [0,1]$ by $u(x) = 2x \cdot \mathbb{I}_{[0,\frac{1}{2})} + (2 - 2x) \cdot \mathbb{I}_{[\frac{1}{2},1]}$, where $\mathbb{I}_A$ denotes the indicator function of the set A.

(a) Show that u is $\mathcal{B}([0,1])/\mathcal{B}([0,1])$ measurable, and determine the image measure $u(\lambda) = \lambda \circ u^{-1}$. (2 pts)

(b) Let $\mathcal{C} = \left\{A \in \mathcal{B}([0,1]) : \lambda\left(u^{-1}(A)\Delta A\right) = 0\right\}$. Show that $\mathcal{C}$ is a σ-algebra. (Note that $u^{-1}(A)\Delta A = \left(u^{-1}(A) \setminus A\right) \dot\cup \left(A \setminus u^{-1}(A)\right)$. (2.5 pts)

(3) Let $(X, \mathcal{A})$ be a measurable space and $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, a partition of X, i.e. $A_n \in \mathcal{A}$ are pairwise disjoint and $X = \dot\bigcup_{n\in\mathbb{N}} A_n$. Consider the function $u : X \to \mathbb{R}$ defined by

$$u(x) = \sum_{j\in\mathbb{N}} 2^j \cdot \mathbb{I}_{A_j}(x).$$

(a) Show that $u \in \mathcal{M}(\mathcal{A})$, i.e. u is $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable. (1.5 pts)

(b) Recall that $\sigma(u) = \left\{u^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\right\}$ is the smallest σ-algebra on X making u Borel measurable. Prove that

$$\sigma(u) = \sigma(\{A_n : n \in \mathbb{N}\}),$$

where $\sigma(\{A_n : n \in \mathbb{N}\})$ is the smallest σ-algebra generated by the countable collection $\{A_n : n \in \mathbb{N}\}$. (2 pts)