Universiteit Utrecht Boedapestlaan 6
Mathematisch Instituut 3584 CD Utrecht

Measure and Integration: Solutions Mid-Term, 2020-21

(1) Let X be a set and μ, ν two **outer measures** on X, i.e. $\mu, \nu : \mathcal{P}(X) \to [0, \infty]$ satisfying the three properties:

(i) $\mu(\varnothing) = \nu(\varnothing) = 0$,

(ii) if $A, B \in \mathcal{P}(X)$ with $A \subseteq B$, then $\mu(A) \le \mu(B)$ and $\nu(A) \le \nu(B)$ (μ and ν are monotone),

(iii) if (A_n) is a sequence in $\mathcal{P}(X)$, then $\mu(\bigcup_n A_n) \le \sum_n \mu(A_n)$ and $\nu(\bigcup_n A_n) \le \sum_n \nu(A_n)$.

Define $\rho : \mathcal{P}(X) \to [0, \infty]$ by $\rho(A) = \max(\mu(A), \nu(A))$. Show that ρ is an outer measure on X, i.e. satisfies properties (i), (ii) and (iii). (2 pts)

Proof: $\rho(\varnothing) = 0$ is immediate since $\max(0, 0) = 0$. The second property is also immediate since if $A \subseteq B$, then $\mu(A) \le \mu(B)$ and $\nu(A) \le \nu(B)$, hence

$$\rho(A) = \max(\mu(A), \nu(A)) \le \max(\mu(B), \nu(B)) = \rho(B).$$

Now let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}(X)$, then

$$\begin{aligned}
\rho\Big(\bigcup_{n=1}^{\infty} A_n\Big) &= \max\Big(\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big), \nu\Big(\bigcup_{n=1}^{\infty} A_n\Big)\Big) \\
&\le \max\Big(\sum_{n=1}^{\infty} \mu(A_n), \sum_{n=1}^{\infty} \nu(A_n)\Big) \\
&\le \sum_{n=1}^{\infty} \max(\mu(A_n), \nu(A_n)) \\
&= \sum_{n=1}^{\infty} \rho(A_n),
\end{aligned}$$

where the second inequality follows from the fact that $\sum_{n=1}^{\infty} \mu(A_n) \le \sum_{n=1}^{\infty} \max(\mu(A_n), \nu(A_n))$, and $\sum_{n=1}^{\infty} \nu(A_n) \le \sum_{n=1}^{\infty} \max(\mu(A_n), \nu(A_n))$. Thus, ρ is an outer measure on X.

(2) Consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the Borel σ-algebra restricted to $[0,1]$ and λ is the restriction of Lebesgue measure on $[0,1]$. Define a map $u : [0,1] \to [0,1]$ by $u(x) = 2x \cdot \mathbb{I}_{[0,\frac{1}{2})} + (2 - 2x) \cdot \mathbb{I}_{[\frac{1}{2},1]}$, where $\mathbb{I}_A$ denotes the indicator function of the set A.

(a) Show that u is $\mathcal{B}([0,1])/\mathcal{B}([0,1])$ measurable, and determine the image measure $u(\lambda) = \lambda \circ u^{-1}$. (2 pts)

(b) Let $\mathcal{C} = \Big\{A \in \mathcal{B}([0,1]) : \lambda\Big(u^{-1}(A)\Delta A\Big) = 0\Big\}$. Show that $\mathcal{C}$ is a σ-algebra. (Note that $u^{-1}(A)\Delta A = \Big(u^{-1}(A) \setminus A\Big) \cup \Big(A \setminus u^{-1}(A)\Big)$. (2.5 pts)

Proof(a): To show u is $\mathcal{B}([0,1])/\mathcal{B}([0,1])$ measurable, it is enough to consider inverse images of intervals of the form $[a, b) \subset [0,1]$. Now,

$$u^{-1}([a,b)) = \Big[\frac{a}{2}, \frac{b}{2}\Big) \dot\bigcup \Big[\frac{2-b}{2}, \frac{2-a}{2}\Big) \in \mathcal{B}([0,1]).$$

Thus, u is measurable. Another quick way of showing measurability is to notice that the functions $x \to 2x$ and $x \to 2 - 2x$ are continuous and hence Borel measurable. Furthermore $\mathbb{I}_{[0,\frac{1}{2})}$ and $\mathbb{I}_{[\frac{1}{2},1]}$

are Borel measurable since $[0,\frac{1}{2}),[\frac{1}{2},1] \in \mathcal{B}([0,1])$. Since products and sums of measurable functions are measurable, we see that u is also Borel measurable.

We claim that $u(\lambda) = \lambda$. To prove this, we use Theorem 5.7. Notice that $\mathcal{B}([0,1])$ is generated by the collection $\mathcal{G} = \{[a,b] : 0 \le a \le b \le 1\} \cup \{\varnothing\}$ which is closed under finite intersections. Now,

$$\begin{aligned} u(\lambda)([a,b]) &= \lambda(u^{-1}([a,b])) \\ &= \lambda([\frac{a}{2},\frac{b}{2}]) + \lambda([\frac{2-b}{2},\frac{2-a}{2}]) \\ &= b-a = \lambda([a,b]). \end{aligned}$$

Since the constant sequence $([0,1])$ is exhausting, belongs to $\mathcal{G}$ and $\lambda([0,1]) = u(\lambda([0,1]) = 1 < \infty$, we have by Theorem 5.7 that $u(\lambda) = \lambda$.

Proof(b): We check the three conditions for a collection of sets to be a σ-algebra. Firstly, the empty set $\varnothing \in \mathcal{B}([0,1])$ and $u^{-1}(\varnothing) = \varnothing$, hence $\lambda\Big(u^{-1}(\varnothing)\Delta\varnothing\Big) = \lambda(\varnothing) = 0$, so $\varnothing \in \mathcal{C}$. Secondly, Let $A \in \mathcal{C}$, then $\lambda\Big(u^{-1}(A)\Delta A\Big) = 0$. We have

$$\lambda\Big(u^{-1}(A)^c\Delta A^c\Big) = \lambda\Big(u^{-1}(A)\Delta A\Big) = 0,$$

with $A^c \in \mathcal{B}([0,1])$, hence $A^c \in \mathcal{C}$. Thirdly, let (A_n) be a sequence in $\mathcal{C}$, then $A_n \in \mathcal{B}([0,1))$ and $\lambda\Big(u^{-1}(A)_n\Delta A_n\Big) = 0$ for each n. Since $\mathcal{B}([0,1])$ is a σ-algebra, we have $\bigcup_n A_n \in \mathcal{B}([0,1])$, and

$$u^{-1}(\bigcup_n A_n) = \bigcup_n u^{-1}(A_n).$$

An easy calculation shows that

$$u^{-1}(\bigcup_n A_n)\Delta\bigcup_n A_n \subseteq \bigcup_n \Big(u^{-1}(A_n)\Delta A_n\Big).$$

By σ-subadditivity and monotonicity of λ, we have

$$\lambda\Big(u^{-1}\Big(\bigcup_m A_m)\Delta\bigcup_n A_n\Big)\Big) \le \lambda\Big(\bigcup_n\Big(u^{-1}(A_n)\Delta A_n\Big)\Big) \le \sum_n \lambda\Big(u^{-1}(A_n)\Delta A_n\Big) = 0.$$

Thus, $\bigcup_n A_n \in \mathcal{C}$. This shows that $\mathcal{C}$ is a σ-algebra.

(3) Let $(X,\mathcal{A})$ be a measurable space and $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, a partition of X, i.e. $A_n \in \mathcal{A}$ are pairwise disjoint and $X = \dot{\bigcup}_{n\in\mathbb{N}} A_n$. Consider the function $u : X \to \mathbb{R}$ defined by

$$u(x) = \sum_{j\in\mathbb{N}} 2^j \cdot \mathbb{I}_{A_j}(x).$$

(a) Show that $u \in \mathcal{M}(\mathcal{A})$, i.e. u is $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable. (1.5 pt)

(b) Recall that $\sigma(u) = \Big\{u^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\Big\}$ is the smallest σ-algebra on X making u Borel measurable. Prove that

$$\sigma(u) = \sigma(\{A_n : n \in \mathbb{N}\}),$$

where $\sigma(\{A_n : n \in \mathbb{N}\})$ is the smallest σ-algebra generated by the countable collection $\{A_n : n \in \mathbb{N}\}$. (2 pts)

Proof(a): For each $n \in \mathbb{N}$, define $u_n(x) = \sum_{j=1}^{n} 2^j \cdot \mathbb{I}_{A_j}(x)$. Since $A_j \in \mathcal{A}$, we see that $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence of non-negative measurable simple functions, i.e. $u_n \in \mathcal{E}^+(\mathcal{A}) \subseteq \mathcal{M}^+(\mathcal{A})$, and

$$u(x) = \lim_{n\to\infty} u_n(x) = \sup_{n\in\mathbb{N}} u_n(x).$$

By Corollary 8.10, it follows that $u \in \mathcal{M}(\mathcal{A})$, i.e. is $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable.

Proof(b): Note that $A_n = u^{-1}\Big(\{2^n\}\Big)$ and $\{2^n\} \in \mathcal{B}(\mathbb{R})$. Hence, $A_n \in \sigma(u)$ for all $n \in \mathbb{N}$, implying that $\sigma(\{A_n : n \in \mathbb{N}\}) \subseteq \sigma(u)$. We now prove the reverse containment. Let $C \in \sigma(u)$, then

$C = u^{-1}(B)$ for some Borel set $B \in \mathcal{B}(\mathbb{R})$. Set $N(B) = \{n \in N : 2^n \in B\}$ By definition of u, we see that

$$C = u^{-1}(B) = \dot{\bigcup_{n \in N(B)}} A_n \in \sigma(\{A_n : n \in \mathbb{N}\}).$$

This shows that $\sigma(u) \subseteq \sigma(\{A_n : n \in \mathbb{N}\})$ and hence $\sigma(u) = \sigma(\{A_n : n \in \mathbb{N}\})$.