# Measure and Integration: Final Exam 2020-21

(1) Let $(X, \mathcal{A}, \mu)$ be a measure space and $u \in \mathcal{L}^1(\mu)$. Define the mesure $\nu$ on $\mathcal{A}$ by $\nu(A) = \int_A |u|\, d\mu$. Prove that for any $v \in \mathcal{L}^1(\nu)$, one has

$$\int v\, d\nu = \int |u| v\, d\mu.$$

(1.5 pts)

(2) Consider the measure space $\big((0,1), \mathcal{B}((0,1)), \lambda\big)$, where $\mathcal{B}((0,1))$ is the Borel $\sigma$-algebra restricted to the interval $(0,1)$ and $\lambda$ is the restriction of Lebesgue measure to $(0,1)$. Let $u \in \mathcal{L}^2(\lambda)$ be **non-negative** and **monotonically increasing**.

(a) Prove that for any $x \in (0,1)$, $\inf\limits_{n \geq 1} u(x^n) = \inf\limits_{y \in (0,1)} u(y)$. (0.5 pt)

(b) Let $w_n(x) = x \cdot u(x^n)$, $n \geq 1$. Prove that $w_n \in \mathcal{L}^2(\lambda)$ for all $n \geq 1$, and that $\lim\limits_{n \to \infty} \|w_n(x)\|_2 = \inf\limits_{y \in (0,1)} u(y) \cdot \frac{\sqrt{3}}{3}$. (2 pts)

(c) Prove that $\lim\limits_{n \to \infty} \int_{(0,1)} x^n e^{x/n} u(x)\, d\lambda(x) = 0$. (1 pt)

(3) Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 < p < \infty$. Suppose $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$ with $\|u_n\|_p \leq \frac{1}{2p+1}$ for $n \geq 1$. Prove that $\left| \sum\limits_{n=1}^{\infty} \left( \frac{u_n}{n} \right)^p \right| < \infty$ $\mu$ a.e. (2 pts)

(4) Consider the product space $\big([1,2] \times [0,\infty), \mathcal{B}([1,2]) \otimes \mathcal{B}([0,\infty)), \lambda \times \lambda\big)$, where $\lambda$ is Lebesgue measure restricted to the appropriate space. Consider the fuction $f : [1,2] \times [0,\infty) \to [0,\infty)$ defined by $f(x,t) = e^{-2xt} \mathbb{I}_{(0,\infty)}(t)$.

(a) Prove that $f \in \mathcal{L}^1(\lambda \times \lambda)$. (2 pts)

(b) Prove that $\int_{(0,\infty)} (e^{-2t} - e^{-4t}) \frac{1}{t} d\lambda(t) = \ln(2)$. (1 pt)